The Incompatibility between Euclidean Geometry and the Algebraic Solutions of Geometric Problems

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***Abstract* — The transition from the "early-modern" mathematical and scientific norms of establishing conventional Euclidean geometric proofs has experienced quite mixed modes of reasoning. For instance, a careful investigation based on the continued attempts by different practitioners to resolve the geometric trisectability of a plane angle suggests serious hitches with the established algebraic angles non-trisectability proofs. These faults found the root for the difficult geometric question about having straightedge and compass proofs for either the trisectability or the non-trisectability of angles. One of the evident gaps regarding the norms for establishing the Euclidean geometric proofs concerns the incompatibility between the smugly asserted algebraic-geometric proofs and the desired inherent Euclidean geometric proofs. We consider an algebraically translated proof of the geometric angle trisection scheme proposed by [1]. We assert and prove that there is a complete incompatibility between the geometric and the algebraic methods of proofs, and hence the algebraic methods should not be used as authoritative means of proving Euclidean geometric problems. The paper culminates by employing the incompatibility proofs in justifying the independence of the Euclidean geometric system.**

***Keywords* — Analytic geometry; Angle-trisection; Constructability; Euclidean geometry; Magnitude; Taylor-expansion.**

I. Introduction

The axiomatization of the Euclidean geometric system outline the principles with plane geometric nature, such that any theorem or demonstration of straightedge-compass geometry may be established with the production of a geometric figure [1], [2]. Intuitively, Euclid also appealed to the common insights of angles [3], [4] while building his idea of geometric magnitudes. In this paper, we look into the resolution of a plane angle as a geometric magnitude. There are principal components of the plane geometric quantities of Euclidean geometry that could go away from our belief of a plane angle. First, in geometry, only a few of the capabilities of plane figures are theoretically applicable. For example, Euclid introduces axioms bearing on angles, wherein he factors to the right angle as a unique figure; however he does not introduce the definitions associated with the geometries of a right angle, along with a definition of horizontal or vertical edges (perpendicular lines) [5]. In this sense, we will slightly assume that the plane geometric representations of an angle lacks sufficient Euclidean axiomatization (as it is considered in the modern perspective (we consider the modern perspective begins slightly from $1600s$)) and thus paving ways for other anti-straightedge and compass geometric approaches. Nonetheless, we consider in the same assumption that the introduction of these non-plane geometric representations of plane angles inherently instantiate invariance (a careful observation made in the development of geometry in the early-modern era (up to the time of Rene' Descartes invention of possible geometric solutions, $1600s$) [6], [7]) with the aid of using tools that are not theoretically applicable to formal Euclidean geometry. Second, we consider that a number of the principles of geometry transcend plane belief with the aid of using their very definition. Hence, Euclid's axioms introduce best principles whose extension to different geometric magnitudes is both infinitely small and large, extending past the boundaries of our belief. For example, for Euclid, a line is a quantity so infinitely thin and infinitely long. These two schools of opinion play a major role in establishing the inherent distinction between the plane Euclidean geometric system and the non-conventions methods of plane geometric proofs. The question of the incompatibility between geometric methods of proof and the algebraic methods of proof stems from the time of Rene' Descartes [8]. One of critical methodological concerns for early-modern geometers is the subject of geometric exactness. "Exactness" often meant establishing suitable standards for determining which curves, viewed as instruments for problem-solving, should be admitted in geometry and which should not [9], [10]. Two fundamental modes were used by ancient and early-modern geometers to define geometrical objects [11]. One method was to describe the construction of a mathematical object, such a curve, in order to define it.

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According to early modern custom, this method of identification was known as "description" (descriptio) or "specification by genesis" [12], [13]. Second, through a property that a mathematical entity, such as a curve, was required to have in order to be a locus, a mathematical entity could be recognized. This is known as "specification by property" [10], [13], [14]. The notion of geometric exactness between algebraic methods and Euclidean geometric norms of proof will as well, form an ingredient in establishing the aimed incompatibility between the two norms of proofs for geometric problems. We thus, respond to the incompatibility question by providing the inherent characteristics of the modes of geometric proofs made from the two contrasting subjects. We hope to write off this discussion by showing that the asserted non-conventional Euclidean geometric methods of proofs imposed to Euclidean geometry lack the rigorous inherent geometric sophistication, to be employed as authoritative methods for establishing geometric proofs. We then conclusively show that algebraic methods of proof cannot qualify to be geometric methods of proof [15].

II. Algebraic Analysis of The "AG-Algorithm" Solution

This analysis is motivated by discussions between the named authors (they retain skepticism about the validity of the established non-geometric angles non-trisectability proofs) two mathematics professors (their names and institutions not to be mentioned here) who weakly hold the opposite view that the established non-geometric algebraic proofs are valid to establish Euclidean geometric statements. We name the operations in this section as "*geometric analysis*" as they do not involve the description of inherent geometric constructions, but rather, the section concerns the analysis of a provided Euclidean geometric scheme based on non-geometric methods. The "***AG-Algorithm***" concerns a scheme for resolving plane angles, and so is the trisectability of an arbitrary angle. The analysis is based on the hypothesized assumption that there is a point of convergence upon which the trisection error of chord $\overline{JB}$ corresponds to the trisection error of the curve $\widehat{JB}$ such that $\angle PAQ = {}^{\angle JAB}/_{3}$ (without elegantly defining the equality here), implying $\overline{PQ} = \overline{BR} = \overline{RS} = \overline{SJ}$ and $\widehat{PQ} = \widehat{BR} = \widehat{RS} = \widehat{SJ}$, since the point S is the resulting "***angle trisection point***" [15]. The aim here is to expose the limitations of using the constructability of a specific angle as a reasonable method of establishing a plane geometric generic impossibility statement. The subsequent workflow concerns investigating the assumption that $\angle PAQ = {}^{\theta}/_{3}$, where $\theta = \angle JAB$. Although this is quite a weak assumption considering the realm of the plane angles, provable conclusion is made regarding the primary goal of the paper. We consider Fig. 1 obtained following the "***AG-Algorithm***" of an arbitrary angle $\angle JAB$ constructed in a unit circle with; $A = (0,0)$, $B = (1,0)$, and, $J = (Cos\theta, Sin\theta)$. Let the points O and N trisect the chord $\overline{JB}$ such that $|BO| = |ON| = |NJ|$, as established in [15]. According to the "***AG-Algorithm***" algorithm, $D' = \left(-{}^{1}/_{2}, 0\right)$. From Fig. 1, we let P and Q be points on the unit circle defined by the following equations:

$P = \ Intersection\ of\ the\ line\ \overline{D'N}\ and\ the\ arc\ \overline{JB}.$

$Q = Intersection\ of\ the\ line\ \overline{OA}\ and\ the\ arc\ \overline{JB}.$

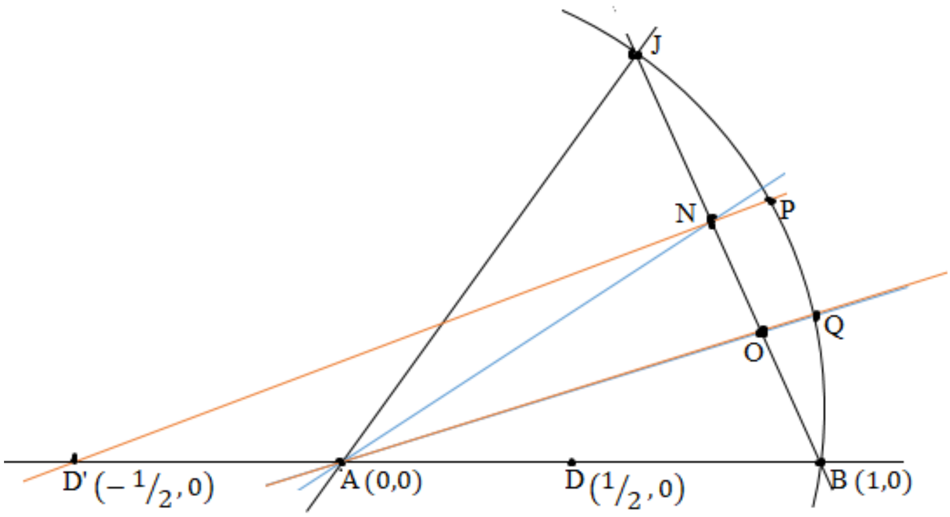

Fig. 1. "AG-Algorithm" Analysis Scheme.

We begging the investigation with the trisection of chord $\overline{JB}$, which aids us in determining the points O and N on the chord $\overline{JB}$. We then use points O and N to construct points P and Q. We assume some period t from the center of the unit circle and set the chord;

$$\overline{JB} = (1,0) + t(Cos\theta - 1, Sin\theta); \quad \text{for } 0 \leq t \tag{1}$$

From equation (1), when $t = 0$ we obtain $B = (1,0)$, and when $t = 1$ we get $J = (Cos\theta, Sin\theta)$. We preserve the points B and J of the unit circle for later use.

Further, assuming that chord $\overline{JB}$ is exactly trisected, we now set $t = {}^2\!/_3$ and solve for point N as:

$$N = (1,0) + {}^2\!/_3\,(Cos\theta - 1, Sin\theta) = \left(1 + {}^2\!/_3\,Cos\theta - {}^2\!/_3, {}^2\!/_3\,Sin\theta\right) = \left({}^1\!/_3 + {}^2\!/_3\,Cos\theta, {}^2\!/_3\,Sin\theta\right).$$

Hence, $N = \left({}^1\!/_3 + {}^2\!/_3\,Cos\theta, {}^2\!/_3\,Sin\theta\right)$ (2)

Again from equation (1), we put $t = {}^1\!/_3$ and solve for point O as follows:

$$O = (1,0) + {}^1\!/_3\,(Cos\theta - 1, Sin\theta) = \left(1 + {}^1\!/_3\,Cos\theta - {}^1\!/_3, {}^1\!/_3\,Sin\theta\right) = \left({}^2\!/_3 + {}^1\!/_3\,Cos\theta, {}^1\!/_3\,Sin\theta\right).$$

So, $O = \left({}^2\!/_3 + {}^1\!/_3\,Cos_\theta, {}^1\!/_3\,Sin\theta\right)$ (3)

This proof assumes that since $\overline{D'N}$ and $\overline{OA}$ are parallel as established by [15] and that since points O and N were obtained using similar circles, chord $\overline{JB}$ is exactly trisected. Though this assumption is practically shown to be an incorrect hypothesis through analysis, (appendices (1) and (2 (b)) in [15]) this paper preserves the hypothesis with the focus to examining the trisection of angles on anti-Euclidean methods.

Case 1: "AG-Algorithm" Solution Error Analysis

Now to compute the angle trisection errors, we have to determine the points P and Q on the circumference of a unit circle. We proceed using points O and N in writing the line equations $\overline{D'N}$ and $\overline{OA}$ as follows:

$$\text{Line } \overline{D'N} \rightarrow y - 0 = \left(\left({}^2\!/_3\,Sin\theta - 0\right) \Big/ \left({}^1\!/_3 + {}^2\!/_3\,Cos\theta + {}^1\!/_2\right)\right)\left(x + {}^1\!/_2\right) \tag{4}$$

Equation (4) further implies that $y = a\left(x + {}^1\!/_2\right)$ (5)

$$\text{Line } \overline{OA} \rightarrow y - 0 = \left(\left({}^1\!/_3\,Sin\theta - 0\right) \Big/ \left({}^1\!/_3 + {}^2\!/_3\,Cos\theta + {}^1\!/_2\right)\right)(x) \tag{6}$$

Equation (6) further implies that $y = b(x)$ (7)

To find the coordinates of the points P and Q we determine the points of intersections between lines $\overline{D'N}$ and $\overline{OA}$ with the unit circle defined as: $x^2 + y^2 = 1$. (8)

Using the equation for $\overline{D'N}$ in (8) we get;

$$x^2 + a\left(x + {}^1\!/_2\right) = 1 \tag{9}$$

Simplifying (9) we obtain equations (10) and (11) as follows.

$$x^2 + a^2\left(x^2 + x + {}^1\!/_4\right) = 1$$

$$(1 + a^2)x^2 + a^2x + {}^{a^2}\!/_4 - 1 = 0$$

$$x = \frac{-a^2 + \sqrt{a^4 - 4(1 + a^2)\left({}^{a^2}\!/_4 - 1\right)}}{2(1 + a^2)} = \frac{-a^2 + \sqrt{a^4 - 4\left({}^{a^2}\!/_4 + {}^{a^4}\!/_4 - 1 - a^2\right)}}{2(1 + a^2)}$$

$$x = \frac{\sqrt{3a^4+4}-a^2}{2(1+a^2)} \tag{10}$$

$$y = a\left(\frac{\sqrt{3a^4+4}-a^2}{2(1+a^2)} + \frac{(1+a^2)}{2(1+a^2)}\right) = a\left(\frac{\sqrt{3a^4+4}+1}{2(1+a^2)}\right) \tag{11}$$

Consequently, equations (10) and (11) give us the point P, according to equation (12)

$$P = \left(\frac{\sqrt{3a^2+4}-a^2}{2(1+a^2)}, \left(\frac{\sqrt{3a^4+4}+1}{2(1+a^2)}\right)\right) \tag{12}$$

Now we find the point of intersection between the line $\overline{OA}$ and the unit circle circumference as follows:

$$x^2 + (bx)^2 = 1 = (1+b^2)x^2 = 1 = x = {}^{1}\!/_{\sqrt{1+b^2}} \tag{13}$$

From equation (13) we make:

$$y = b(x) = {}^{b}\!/_{\sqrt{1+b^2}} \tag{14}$$

So that the point Q becomes:

$$Q = \left({}^{1}\!/_{\sqrt{1+b^2}}, {}^{b}\!/_{\sqrt{1+b^2}}\right), \text{ which lies on the unit circle.} \tag{15}$$

We set up the point J on the unit circle as shown in equation (16).

$$J = (Cos\theta, Sin\theta) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \tag{16}$$

So that arbitrarily, we obtain Pythagorean triangles of the form in equation (17).

$$t = {}^{a}\!/_{b} \Rightarrow \left(\frac{1-\left(\frac{a}{b}\right)^2}{1+\left(\frac{a}{b}\right)^2}, \frac{2\left(\frac{a}{b}\right)}{1+\left(\frac{a}{b}\right)^2}\right) \Rightarrow \left(\left(1-\left(\frac{a}{b}\right)^2\right)^2 + 2\left(\frac{a}{b}\right)^2 = \left(1+\left(\frac{a}{b}\right)^2\right)^2\right) \tag{17}$$

After some investigations for different t values, we randomly set $t = \frac{1}{181}$ to operate within the confines of the "***AG-Algorithm***".

It follows from equation (17) that: $\left(\frac{1-\frac{1}{181^2}}{1+t^2}, \frac{2\left(\frac{1}{181}\right)}{1+\frac{1}{181^2}}\right) = \left(\frac{181-1}{181^2+1}, \frac{2(181)}{181^2+1}\right) = \left(\frac{32760}{32762}, \frac{2(181)}{32762}\right) = \left(\frac{16380}{16381}, \frac{181}{16381}\right)$ (18)

So that we obtain the Pythagorean equivalence; $181^2 + 16380^2 = 16381^2$ in which both 181, 16, 381 are prime numbers. This completely brings in the notion of the rational number theorem in the computation.

Observe $Cos\theta = \frac{16380}{16381} \Rightarrow \theta = 0.633096°$, thus $Cos\theta < 0.75°$.

We now investigate for a and b, as follows;

$$a = \text{ slope of } \overline{D'N} = \frac{45}{5+4c} = \frac{4\frac{2t}{1+t^2}}{5+4\frac{1-t^2}{1+t^2}} = \frac{\frac{8t}{(1+t^2)}}{\frac{5(1+t^2)+4(1-t^2)}{1+t^2}} = \frac{8t}{9+t^2} \tag{19}$$

$$b = \text{ slope of } \overline{A0} = \frac{5}{2+c} = \frac{\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} = \frac{\frac{2t}{(1+t^2)}}{\frac{2(1+t^2)+1-t^2}{1+t^2}} = \frac{2t}{3+t^2} \tag{20}$$

Now we have to determine the point P which is the point of intersection of the line $\overline{\mathrm{D'N}}$ with the unit circle. Using the previous formula for P in terms of a

$$P = \left(\frac{\sqrt{3a^2+4}-a^2}{2(1+a^2)}, \frac{a\sqrt{3a^2+4}+a}{2(1+a^2)}\right)$$

we obtain;

$$P=\left(\frac{\sqrt{3\left(\frac{8t}{9+t^2}\right)^2+4}-\left(\frac{8t}{9+t^2}\right)^2}{2\left(1+\left(\frac{8t}{9+t^2}\right)^2\right)},\frac{\frac{8t}{9+t^2}\sqrt{3\left(\frac{8t}{9+t^2}\right)^2+4}+\frac{8t}{9+t^2}}{2\left(1+\left(\frac{8t}{9+t^2}\right)^2\right)}\right)$$

$$P=\left(\frac{\frac{9+t^2}{9+t^2}\frac{\sqrt{3(8t)^2+4(9+t^2)^2}}{9+t^2}-\frac{(8t)^2}{(9+t^2)^2}}{2\frac{\left((9+t^2)^2+(8t)^2\right)}{(9+t^2)^2}},\frac{\frac{8t}{9+t^2}\frac{\sqrt{3(8t)^2+4(9+t^2)}}{9+t^2}+\frac{8t(9+t^2)}{(9+t^2)^2}}{2\frac{(9+t^2)^2+(8t)^2}{(9+t^2)^2}}\right)=$$

$$\left(\frac{9+t^2\sqrt{192t^2+4(81+t^4+18t^2)}-64t^2}{2(81+t^4+18t^2+64t^2)},\frac{8t\sqrt{4(48t^2+81+t^4+18t^2\ )}+2\times4t(9+t^2)}{2(81+t^4+18t^2+64t^2)}\right)$$

$$P=\left(\frac{9+t^2\sqrt{81+66t^2+t^4}-32t^2}{81+82t^2+t^4},\frac{8t\sqrt{81+66t^2+t^4}+4t(9+t^2)}{81+82t^2+t^4}\right) \tag{21}$$

Let's now determine Q as the intersection of $\overline{A0}$ with the unit circle. We consider;

$Q=\left(\frac{1}{\sqrt{1+b^2}},\frac{b}{\sqrt{1+b^2}}\right)$ Where; $b=\frac{2t}{3+t^2}$

By inserting b into:

$$Q=\left(\frac{1}{\sqrt{1+\left(\frac{2t}{3+t^2}\right)^2}},\frac{\frac{2t}{3+t^2}}{\sqrt{1+\left(\frac{2t}{3+t^2}\right)^2}}\right)=\left(\frac{3+t^2}{\sqrt{(3+t^2)^2+(2t)^2}},\frac{\frac{2t}{3+t^2}}{\frac{\sqrt{(3+t^2)^2+(2t)^2}}{3+t^2}}\right)=\left(\frac{3+t^2}{\sqrt{9+6t^2+t^4 4t^2}},\frac{2t}{\sqrt{9+6t^2+t^4 4t^2}}\right)$$

$$Q=\left(\frac{3+t^2}{\sqrt{9+10t^2+t^4}},\frac{2t}{\sqrt{9+10t^2+t^4}}\right) \tag{23}$$

From Fig. 1, we apply the dot product in determining the angle between $\overline{AQ}$ and $\overline{AP}$ as follows:

$$\underline{u}\cdot\underline{v}=|\underline{u}|\cdot|\underline{v}|\cdot \mathrm{Cos}\,(u,v) \tag{24}$$

Since P and Q are on the unit circle we get:

$$Cos\ (\angle PAQ)=\overrightarrow{AP}\cdot\overrightarrow{AQ} \tag{25}$$

To determine the dot product, we consider the trigonometric functions (*cosine* and *sine*) for an angle of some magnitude at some accuracy and the variables a and b as follows.

$$Cos(\angle PAQ)=\overrightarrow{AP}\cdot\overrightarrow{AQ}=\left\{\left(\frac{9+t^2\sqrt{81+66t^2+t^4}-32t^2}{81+82t^2+t^4}\right).\left(\frac{3+t^2}{\sqrt{9+10t^2+t^4}}\right)\right\}+\left\{\left(\frac{8t\sqrt{81+66t^2+t^4}+4t(9+t^2)}{81+82t^2+t^4}\right).\left(\frac{2t}{\sqrt{9+10t^2+t^4}}\right)\right\} \tag{26}$$

At least both expressions have the same denominator and are expressed as t^2.

$$Cos\ (\angle PAQ)=\overrightarrow{AP}\cdot\overrightarrow{AQ}=\frac{\sqrt{81+66t^2+t^4}[(9+t^2)+(3+t^2)+16t^2]-[32t^2(3+t^2)-4t(9+t^2).2t]}{81+82t^2+t^4\,.\,\sqrt{9+10t^2+t^4}}$$

$$Cos\ (\angle PAQ)=\overrightarrow{AP}\cdot\overrightarrow{AQ}=\frac{\sqrt{81+66t^2+t^4}[27+12t^2+t^4 16t^2]-[96t^2+32t^4-72t^2-8t^4]}{81+82t^2+t^4\,.\,\sqrt{9+10t^2+t^4}}$$

$$Cos\ (\angle PAQ)=\overrightarrow{AP}\cdot\overrightarrow{AQ}=\frac{\sqrt{81+66t^2+t^4}[27+28t^2+t^4]-[24t^2+24t^4]}{81+82t^2+t^4\,.\,\sqrt{9+10t^2+t^4}} \tag{27}$$

We seek to simplify equation (27) using factorization method.

$$t^4 + 82t^2 + 27 = 0 \Rightarrow t^2 = \frac{-28 \pm \sqrt{28^2 - 4.27}}{2} = \frac{-28 \pm 26}{2} = \begin{Bmatrix} -\frac{54}{2} \\ -\frac{2}{2} \end{Bmatrix} = \begin{Bmatrix} -27 \\ -1 \end{Bmatrix}$$

$$t^4 + 82t^2 + 81 = 0 \Rightarrow t^2 = \frac{-82 \pm \sqrt{82^2 - 4.81}}{2} = \frac{-82 \pm 80}{2} = \begin{Bmatrix} -\frac{162}{2} \\ -\frac{2}{2} \end{Bmatrix} = \begin{Bmatrix} -81 \\ -1 \end{Bmatrix}$$

$$Cos\ (\angle PAQ) = \frac{\sqrt{81 + 66t^2 + t^4}(t^2 + 27)(t^2 + 1) - 24t^2(t^2 + 1)}{(81 + t^2)(t^2 + 1)\sqrt{9 + 10t^2 + t^4}}$$

$$Cos\ (\angle PAQ) = \frac{\sqrt{81 + 66t^2 + t^4}(t^2 + 27) - 24t^2}{(81 + t^2)\sqrt{9 + 10t^2 + t^4}}$$

So that $81 + 66t^2 + t^4$ and $9 + 10t^2 + t^4$ is reducible.

A. Establishing the Non-Trisectability of ∠JAB

The goal of the subsequent computations is to illustrate that the $\angle JAB$ is not exactly trisected. Consider claim (1).

$Claim$ 1: It is impossible to trisect $\angle JAB$.

Proof: To show that $\angle JAB$ cannot be trisected we consider $t = \frac{1}{181}$.

Then: $Cos\ (\angle PAQ) = \frac{\sqrt{81(181)^4 + 66(181)^2 + 1}(27(181)^2 + 1) - 24(181)^2}{(81(181^2) + 1)\ \sqrt{9(181)^2 + 10(181)^2 + 1}}$

$$= \frac{\sqrt{86938095028}(884548) - (786264)}{(2653642)\ \sqrt{9659875700}}$$

Next, we perform prime factorization so that;

$$\frac{\sqrt{2^2 \times 19^2 \times 37 \times 1627201} \times 2^2 7^2 \times 4315 - 2^3 \times 3 \times 181^2}{2 \times 1326821 \times \sqrt{2^2 \times 5^2 5897 \times 16381}} = \frac{\sqrt{37 \times 1627201} \times 2 \times 19 \times 7^2 \times 4513 - 2 \times 3 \times 181^2}{1326821 \times 5 \times \sqrt{589716381}}$$

$$= \frac{\sqrt{60162437} \times 8403{,}206 - 196556}{6634105 \times \sqrt{96598757}} = \frac{\sqrt{\mathrm{a}} \times \mathrm{m} - \mathrm{n}}{\mathrm{R}\sqrt{\mathrm{b}}}$$

To determine the trisecting accuracy, we apply the cosine triple angle formula as follows.

$$RHS = 4\ (Cos(\angle PAQ)^3) - 3(Cos(\angle PAQ)) = \frac{\sqrt{\mathrm{a}} \times \alpha - \beta}{\Upsilon \times \sqrt{\mathrm{b}}} \qquad (28)$$

Where:

$\alpha = [4\mathrm{am}^2 + 12\mathrm{n}^2 - 3\mathrm{R}^2\mathrm{b}] \times \mathrm{m}$

$\beta = [12\mathrm{am}^2 + 4\mathrm{n}^2 - 3\mathrm{R}^2\mathrm{b}] \times \mathrm{n}$

Since R^2b is an odd number and $4\mathrm{am}^2 + 12\mathrm{n}^2 - 3\mathrm{R}^2\mathrm{b}$ and $12\mathrm{am}^2 + 4\mathrm{n}^2 - 3\mathrm{R}^2\mathrm{b}$ are even, we note that α and β are non-zero integers. Hence:

$\frac{\mathrm{p}}{\mathrm{q}} = RHS$ means $\frac{\mathrm{p}}{\mathrm{q}} = \frac{\sqrt{\mathrm{a}} \times \alpha - \beta}{\Upsilon \times \sqrt{\mathrm{b}}} \Leftrightarrow \mathrm{p}\Upsilon\sqrt{\mathrm{b}} = \alpha\mathrm{q}\sqrt{\mathrm{a}} - \beta \Rightarrow \mathrm{p}^2\mathrm{q}^2 a + \beta^2 - 2\alpha\beta\mathrm{q}\sqrt{\mathrm{a}}$

$$\sqrt{\mathrm{a}} = \frac{\mathrm{p}^2\mathrm{q}^2 a + \beta^2 - \mathrm{p}^2\Upsilon^2 b}{2\alpha\beta\mathrm{q}} \qquad (29)$$

The expression on the left-hand side is irrational, while the fraction on the right-hand side is rational. Hence the solution leads to a contradiction. So, therefore, the RHS cannot equal $LHS = \frac{1 - \left(\frac{1}{181}\right)^2}{1 + \left(\frac{1}{181}\right)^2}$. Hence, the trisection investigation for $\angle JAB$ fails.

By contradiction, equation (29) shows that the trisection of a particular angle is not possible via straightedge and compass operations. Without scrupulous reasons, this specific view is extended to the generic algebraic way of proving the non-trisectability of angels.

B. Case 2: Further Analysis (An investigation to the AG-Algorithm Accuracy)

This section of proof proceeds from $case$ 1. According to algebra, the angle trisection impossibility is conceived when we encounter a single case of an angle that cannot be trisected. Thus, how to determine if all angles cannot be trisected becomes pure analysis (and even more, there is no single algebraic method on the angles non-trisectability that works for all plane angles). For instance, let us consider the "***AG-Algorithm***" for all angles less than 1°. To do that we return to the expression for $\cos(\angle PAQ)$ and set $t^2 = x$. Then:

$$Cos\ (\angle PAQ) = \frac{\sqrt{81+66x+x^2}(27+x)-24x}{(81+x)\sqrt{9+10x+x^2}} \tag{30}$$

Expanding the right-hand side as a Taylor series with x around $x = 0$ we get

$$Cos(\angle PAQ) = 1 - \frac{2x}{9} + \frac{38}{243}(x^2) - \frac{93}{729}x^3 + \frac{7250}{59049}x^4 \ldots \tag{31}$$

Applying this expansion in solving the formula for RHS we get:

$$RHS = (4 \times \cos(\angle PAQ))^3 - 3 \times \cos(\angle PAQ)$$

$$RHS = 1 - 2x + 2x^2 - \frac{1522}{729}x^3 + \frac{43462}{19683}x^4 \ldots \tag{32}$$

On the other hand, the correct expression of LHS is;

$$LHS\ = \frac{1-t^2}{1+t^2} = \frac{1-x}{1+x} = \frac{1+x-2x}{1+x} = 1 - 2x(1 - x + x^2 - x^3 + x^4 \ldots)$$

$$LHS = 1 - 2x + 2x^2 - 2x^3 + 2x^4 \ldots) \tag{33}$$

So the error in the trisection algorithm for angles becomes;

$$LHS - RHS = \frac{66}{729}.x^3 - \frac{4096}{19683}x^4 = \frac{66}{729}.t^6 - \frac{4096}{19683}t^8$$

$$LHS - RHS = \frac{66}{729}\left(\frac{\sin(\theta)}{1+\cos(\theta)}\right)^6 - \frac{4096}{19683}\left(\frac{\sin(\theta)}{1+\cos(\theta)}\right)^8 \tag{34}$$

From equation (34) we extract the t value as shown in equation (35).

$$\frac{1-t^2}{1+t^2} = cos(\theta) \Leftrightarrow 1 - t^2 = cos(\theta) + (cos(\theta) \times t^2) \Leftrightarrow 1 - cos(\theta) = \ (cos(\theta) \times t^2)$$

$$\Leftrightarrow \frac{1-cos(\theta)}{1+cos(\theta)} = t^2$$

$$\Leftrightarrow t = \sqrt{\frac{1-cos(\theta)}{1+cos(\theta)}} = \sqrt{\frac{(1-cos(\theta))(1+cos(\theta))}{(1+cos(\theta))^2}} = \frac{\sqrt{1-\cos^2(\theta)}}{1+cos(\theta)} = \frac{sin(\theta)}{(1+cos(\theta))} \tag{35}$$

Equation (35) show that as the value for θ decreases from 1° to $\theta > 0°$ the corresponding angle trisection error decreases (appendix 2 (a) [15]). We suppose that if such an error never gets to zero, then the provided angle trisection geometric scheme is generally incorrect. Further, by investigations using a MATLAB code constructed by (appendix 2 (b) [15]), it was shown that as the angle size decreases, the line trisection error is always greater than the "***AG-Algorithm***" angle trisection error.

III. Characterizing The Distinction between The Euclidean Geometric Proofs in Contrast to The Non-euclidean Geometric Proofs

This section aims at revealing the inherent characteristic differences between the Euclidean geometric proofs (interchangeably applied as the conventional proofs) and the mechanical proofs (those involving non-Euclidean geometric proofs) employed in establishing straightedge and compass solutions.

A. Characterizing the Mechanical (Algebraic) Methods of Proof for Geometric Problems

We focus on $case$ 1 and $case$ 2 established in the previous section. Starting with $case$ 1, we examine the algebraic nature of the conclusion made following equation 29. Geometrically, equation 29 is falsely constructed to examine a Euclidean construction that does not have the properties the equation exhibits. The LHS of the equation says that if one constructs the square root of a quantity employing only straightedge and compass operations, the square root does not geometrically equate to any other construction that is not a square root operation (the RHS of the equation). From a Euclidean position, the constructability of a geometric magnitude is not restricted to other scripted conditions (as is the case with algebraic means of establishing solutions) besides the use of the required geometric tools. This unique feature of the Euclidean geometric system makes the algebraic solutions (of the form equation 29) false Euclidean geometric solutions. Inherently, the pathway employed to equation 29 is meant to result in comparing two distinct quantities, and then a contradiction conclusion is constructed. On the opposite, in the Elements, Euclid does not establish such a condition that any constructible geometric quantity should be expressible as a fraction of two integers. We think that within the confines of the plane geometric system, it is by a natural coincidence that the notion of rationality (when considered from the early-modern perspectives) will exist and not inherently from the Euclidean geometric structure. Thus, this form of relaxed (conditional free) Euclidean geometric solutions makes equation 29 a false geometric solution. Considering that in terms of Euclidean geometry the term "*irrational*" is applied as "*incommensurate*" and "*rational*" as "*commensurate*", we assert that equation 29 does not qualify as a Euclidean measure for constructability. Euclid employed comprehensively two ways for examining the constructability of magnitudes; considering the ratios of the similar kind magnitudes, and as well, the ratios of the squares on the constructed magnitudes of similar kinds. One solid application for this Euclid's notion is in the examination of the diagonal of a square. Coining from the Euclidean language one states that, the diagonal of a square is both rational and irrational. Equation 29 does not have that Euclidean notion of testing for the constructability of a geometric magnitude. Further, looking at the general implication of equation 29, one can reasonably rule out that the equation fails as it represents the examination of a particular angle. We say from the Euclidean perspective that the only common property between plane angles is that they are magnitudes bound between two rays, at a vertex. This makes the resolution of angles one of the most difficult exercises in geometry. However, neither does algebra offer alternatively equivalent means of making angles to have a common property. The notion of translating an angle to a straight line segment as suggested through the exposed mathematical formulations fails to meet the desired geometric rigor for examining quantities of a similar kind. Geometrically, there is no straightedge-compass construction for a line segment of magnitude "*zero*" such that the ratio $\cos(\theta) = \left({}^{0}/_{n}\right)$ (with n as a constructed line segment) exists. The ratio $\cos(\theta) = \left({}^{0}/_{n}\right)$ is geometrically invalid thus invalidating the falsely constructed geometric statement, using algebra. Considering $case$ 2, equation 35 does not provide a reasonably equivalent geometric proof that the constructed angle is not exactly as desired. Indeed, equation 35 is completely analytic and its genetic features completely violate the rigor governing Euclidean geometric proofs. In the case of exactness, a careful examination of equation 35 reveals that in all sense, the equation imply analytical approximations as valid geometric measure for geometric exactness. This is consequentially, fatal geometric misconception. The notion of exactness in Euclidean geometric solutions defy the use of any sorts of approximations [10]. Thus, equations 29 and 35 fall off the Euclidean geometric tests, invalidating the complete use of algebraic operations as authoritative geometric solutions.

B. Characterizing the Euclidean Geometric Proofs

The characteristic Euclidean geometric methods of proof are provided by [10], [15]. The sort of tests required for Euclidean geometric solutions does not involve comparing magnitudes of a distinct kind as suggested from equation 29. Inherently, Euclidean geometry involves the formalization of geometry by axioms that directly speak about the fundamental concepts of geometry such as points and lines instead of about a backdrop for such object as non-Euclidean solutions suggest. Thus the approaches established through $case$ 1 and $case$ 2 do not qualify as genetic Euclidean geometric solutions. However, analytically, it is shown [15] that interfacing Euclidean geometry with algebra is reasonable enough, provided that methodic consistency is exhibited throughout the unification practice. Unfortunately, this unification cannot complete without breaking the inherent governing characteristic rules from either of the subjects divide. We consider the characteristic Euclidean geometric proof established by one of the authors of this paper, in [15] ([15], (Fig. 13 and equation 67)). Considerably, (the Fig. 13, in [15]) has offered a general scheme showing that any plane angle can be trisected. If we reject the truth formalized in this scheme, then there will be multiple Euclidean geometric reasons for rejecting any impossibility proof imposed to geometry using algebraic methods. For instance, there is no such opposite generic proof from algebra showing that angles cannot be generally trisected or constructed via straightedge-compass operations. To clarify this, the misconception people have had is that if we exhibit a single case that shows an angle has not been exactly trisected, then the problem becomes unsolvable. This notion exhibits serious faults with the use of algebra as a substitute for Euclidean geometric solutions in that, it does not offer the desired solutions but rather, provides weak distinctive methodic approaches for showing that specific distinct angles cannot be trisected. If we contrast the angles non-trisectability proofs with the proof for the statement of a distinct problem (say the cube duplication problem), we notice that even from the algebraic viewpoint, the cube duplication impossibility proof is generic in the sense that all cubes have completely similar geometric characteristic, thus making a specific proof, validly generic. That is not the case with the algebraic proofs for angles non-trisectability. This problem has been addressed by [15]. Thus, as mentioned in section (A), algebraic proofs do not qualify as genetic Euclidean geometric proofs for the multiple established reasons.

IV. Discussion

Throughout section (III), a comprehensive discussion is provided to show that the characteristic inconsistencies between the Euclidean geometric proof and the non-Euclidean proofs completely separate the two norms of establishing geometric solutions. Thus, in this paper, we have elaborated (equation 29 offers the basic algebraic non-trisectability interpretation) on the critique established by [15] showing that algebra lacks the proper axiomatization to establish Euclidean geometric proof. This paper also contributes to answering the ancient question about the proper methods acceptable for establishing Euclidean geometric proof. Rene' Descartes [8] attempted to work out this question with the use of analytic methods, although he carefully avoided using genetic algebraic approaches for plane geometric problems. We, therefore, conclude that based on the established geometric-algebraic methodic inconsistencies through section (III), the subject of Euclidean geometry is independent enough to be supplemented and supervened with falsely weak algebraic proofs. The lack of a generic algebraic method that shows the non-trisectability of angles exposes the limitations of the non-Euclidean geometric approaches used in establishing plane geometric problems. The genetic nature of Euclidean geometry makes it rich with methods for reducing complex problems (as exposed by [15]) to simpler solvable problems. This paper also offers a turning point to exploring the reasons for different methods showing the non-constructability of specific angles, all summing up to a similar impossibility conclusion, while there is no other such inconsistent proofs in the scientific and mathematics enterprises, as it is the case for angles resolutions.

V. conclusion

Throughout the workflow, attempts have been made to flesh out the inconsistencies between the Euclidean geometric systems and the non-Euclidean geometric system. These inconsistencies suggest the complete incompatibility between the two subjects, thus invalidating the use of algebraic strings as Euclidean geometric proofs. Thus, this paper opens up an opportunity for examining the root problems with the algebraically established geometric proofs, starting with the P.L Wantzel's proof for the angles non-trisectability of 1837. Further, the hope is that scholars will always, look carefully at the gaps in the understanding of the Euclidean geometric system between the minds of up to 17^{th} century, and the sort of Euclidean geometric understanding to the 21^{st} century. This paper asserts that Euclidean geometry has been seriously violated in translating straightedge-compass geometric problems to algebraic (analytic) problems without having a reasonable consideration to what the rigor to follow, for an algebraic statement to be geometrically acceptable.

Conflict of Interest

Authors declare that they do not have any conflict of interest.

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