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A SEMI-PARAMETRIC MULTIPLICATIVE BIAS REDUCTION DENSITY WITH A PARAMETRIC START

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Abstract

In this study, a multiplicative semiparametric fourth order bias reduction density is proposed. The proposed density accommodates all parametric distributions and produces optimum results even in situations where the underlying parametric distributions are not the best approximations to the correct density of the data. This offers an impressive option for estimating statistics with large sample properties

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such as complex indixes. It consists of parametric estimate multiplied by a nonparametric correction function. The simulation results show its practical potential.

1. Introduction

Density estimation has become an important area of study in recent times. This is due to the fact that most statistical estimation approaches use nonparametric estimation, which generally requires enhanced smoothing to improve efficiency and precision of estimates. This calls for consideration of extension of basic kernel density estimators to complicated estimation methods. For instance, variable kernel density estimation (Abramson [1]; Jones [5]), biased reduction density estimation (Hjort and Glad [3]; Jones et al. [6]; Scott [8]; Silverman [10]). All these employed different strategies to bias reduction. Whilst Abramson [1] employed larger bandwidths in low density regions with low bandwidths in high density regions, Hjort and Glad [3]; Jones et al. [6]; Scott [8]; Silverman [10] considered a high degree smoothed function which increased the number of observations used in the estimation procedure, thereby increasing precision and reducing bias. Another important aspect of density estimation is the use of nonparametric and semi-parametric transformation estimation procedures (Abramson [1]; Hjort and Glad [3]; Jones et al. [6]; Ruppert and Cline [7]). This paper extends the work by Hjort and Glad [3] from $O(n^2)$ to $O(n^4)$. This is particularly useful in estimating large sample statistics such as complex indixes.

2. Proposed Density Estimator

In this study, a multiplicative semi-parametric biased reduction density estimator is proposed. The approach is to start with a parametric density estimate and multiply by a nonparametric kernel estimate. The general form of the density is

$$\hat{f}(x) = f(x, \hat{\theta})\hat{r}(x)$$

$$= \frac{1}{n}\sum_{i=1}^{n} K_h(X_i - x)\frac{f(x, \hat{\theta})}{f(X_i, \hat{\theta})}, \tag{1}$$

where the nonparametric correction function is

$$\hat{r}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_h(X_i - x)}{f(X_i, \hat{\theta})}. \tag{2}$$

The $X_i's$ are independent observations from an unknown population density. The parametric estimation used in this estimation process does not have to be necessarily the correct underlying distribution, even when the parametric form is crude, the method works considerably well. Note that the usual kernel density estimator is given by

$$\hat{f}_k(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(X_i - x) \tag{3}$$

with $K_h(z) = h^{-1}K(h^{-1}z)$ and $K(z)$ being the kernel function. In this paper, the kernel function is assumed to be a symmetric probability density with finite variance, $\sigma_k^2 = \int z^2 K(z)dz$ and roughness $R(K) = \int K(z)^2 dz$. The mean and variance are

$$E\hat{f}_k(x) \doteq f(x) + \frac{1}{2}\sigma_k^2 h^2 f''(x) \tag{4}$$

and

$$Var\hat{f}_k(x) \doteq \frac{R(K)f(x)}{nh} - \frac{f(x)^2}{n}. \tag{5}$$

For details and current state of research in this area, see (Cameron and Trivedi [2]; Hjort and Glad [3]).

3. Nonparametric Correction with a Fixed Start

Given f_0, a fixed density, presumably a guess estimate of *f*. Let

$$f = f_0 r. \tag{6}$$

The intent is to estimate the nonparametric correction factor r using kernel smoothing. A possible representation is

$$\begin{aligned}\hat{f}(x) &= f(x, \hat{\theta})\hat{r}(x)\\ &= \frac{1}{n}\sum_{i=1}^{n} K_h(X_i - x)\frac{f(x, \hat{\theta})}{f(X_i, \hat{\theta})}\end{aligned} \tag{7}$$

with

$$\hat{r}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{K_h(X_i - x)}{f(X_i, \hat{\theta})}. \tag{8}$$

Using a fixed density, $f_0(x)$ such as the uniform distribution gives the ordinary kernel estimator, with the mean

$$\begin{aligned}E\hat{r}(x) &= \int K(z)r(x + hz)dz\\ &= r(x) + h\mu_1 r'(x) + \frac{1}{2}h^2\mu_2 r''(x) + \frac{1}{3!}h^3\mu_3 r'''(x)\\ &\quad + \frac{1}{4!}h^4\mu_4 r^{(iv)}(x) + O(h^4)\end{aligned} \tag{9}$$

$$\Rightarrow E\hat{r}(x) = r(x) + \frac{1}{24}h^4\mu_4 r^{(iv)}(x) + O(h^4) \tag{10}$$

and bias

$$bias(\hat{r}(x)) = \hat{r}(x) - r(x) \tag{11}$$

$$= \frac{1}{24}h^4\mu_4 r^{(iv)}(x) + O(h^4). \tag{12}$$

(13)

It has the variance

$$\operatorname{var}(\hat{r}(x)) = \frac{1}{nh} E\left(K\left(\frac{X_i - x}{h}\right)\right)^2 - \frac{1}{n}\left(\frac{1}{h} E\left(K\left(\frac{X_i - x}{h}\right)\right)^2\right) \tag{14}$$

$$\operatorname{var}(\hat{r}(x)) = \frac{1}{nh}\int_{-\infty}^{\infty} K(z)^2 r(x)dz + O\left(\frac{h^4}{n}\right) - \frac{r(x)^2}{n} \tag{15}$$

$$= \frac{r(x)R(K)}{nh} - \frac{r(x)^2}{n} + O\left(\frac{h^4}{n}\right) \tag{16}$$

with $R(K) = \int_{-\infty}^{\infty} K(z)^2 dz$ as the roughness of the kernel.

The proposed estimator has an expectation of

$$E(\hat{f}(x)) = \int_{-\infty}^{\infty} K(z) f(x + hz)dz \tag{17}$$

$$= \int_{-\infty}^{\infty} K(z)\left(f(x) + \frac{1}{24} h^4 \mu_4 f^{(iv)}(x) + O(h^4)\right)dz \tag{18}$$

$$= f(x) + \frac{1}{24} h^4 \mu_4 f^{(iv)}(x) + O(h^4) \tag{19}$$

and bias

$$\Rightarrow E(\hat{f}(x)) - f(x) = \frac{1}{24} h^4 \mu_4 f^{(iv)}(x) + O(h^4) \tag{20}$$

$$= \frac{1}{24} h^4 \mu_4 f_0(x) r^{(iv)}(x) + O(h^4) \tag{21}$$

since $f(x) = f_0 r = f_0(x)r(x)$.

The variance of the proposed estimator is

$$\operatorname{var}(f(x)) = \frac{R(K)}{nh} f(x) - \frac{f(x)^2}{n}. \tag{22}$$

From the foregoing, it is clear that, the variance of the proposed estimator is of the same size to that of the traditional kernel estimator according to the approximation order used and same bias similar to the same order $O(h^4)$ but directly proportional to $f_0 r^{(iv)}$ instead of $f^{(iv)}$. Since mostly $f_0 \leq 1$, it affords a smaller bias than ordinary kernel estimation in most cases. In cases where f_0 is a good guess, r is expected to near constant and $r^{(iv)}$ very small, describing some neighborhood of densities around f_0, where the proposed method is better than the traditional kernel.

3.1. Nonparametric correction on a parametric start

Given a parametric family of densities having a multidimensional parameter $\theta = (\theta_1, \ldots, \theta_p)'$ belonging to a given open and connected region in p-space. Suppose $f(x, \hat{\theta})$ is the parametric start and $\hat{\theta}$, a maximum likelihood estimator. Let $f(x, \hat{\theta})$ be an estimated normal density. The description of data summary given is not intended to obtain the true underlying density; the proposed method works well in cases where f cannot be well approximated by any given $f(\cdot, \theta)$. The critical issue is to obtain the right correction function $f(x)/f(x, \theta)$ using kernel smoothing. Thus,

$$\hat{f}(x) = f(x, \hat{\theta}) \frac{1}{n} \frac{\sum_{i=1}^{n} K_h(X_i - x)}{f(X_i, \hat{\theta})}. \tag{23}$$

The maximum likelihood estimator seeks θ_0, the minimum false value by the Kullback-Leibler distance measure

$$\int f(x) \log\{f(x)/f(x_i, \theta)\} dx \tag{24}$$

from the true density, f to an approximant $f(\cdot, \theta)$. If $f_0(x) = f(x, \theta_0)$ for the given best parametric approximant, and $u_0 = \frac{\partial}{\partial\theta} \log f(x, \hat{\theta}_0)$, the score function evaluated at this parameter value. Using Taylor series expansion,

$$\frac{f(x, \hat{\theta})}{f(X_i, \hat{\theta})} = \exp\{\log f(x, \hat{\theta}) - \log f(X_i, \hat{\theta})\} \tag{25}$$

$$\doteq \frac{f_0(x)}{f(X_i)} + \frac{f_0(x)}{f(X_i)}\{u_0(x) - u_0(X_i)\}'(\hat{\theta} - \theta) \tag{26}$$

which implies

$$\hat{f}(x) \doteq \frac{1}{n}\sum K_h(X_i - x)\frac{f_0(x)}{f_0(X_i)}[1 - \{u_0(X_i) - u_0(x)\}'(\hat{\theta} - \theta)] \tag{27}$$

$$= f^*(x) + V_n(x). \tag{28}$$

Thus, $f^*(x)$ is the same as the original proposed estimator, except that f_0 function here is not unique and the $V_n(x)$ represents the parametric estimation variance.

Expressing $\hat{\theta} - \theta_0$ as an average of *iid* zero mean variables and the remainder terms gives the approximate bias and variance of $\hat{f}(x)$.

For regular estimators that has an influence with finite covariance matrix, suppose F is the true distribution, the cumulative function of *f*, and F_n, the empirical distribution function. The functional estimators for θ in the form of $\hat{\theta} = T(F_n)$ with influence function

$$I(x) = \lim_{\varepsilon \to 0} \{T((1 - \varepsilon)F + \varepsilon\delta_x) - T(F)\}/\varepsilon.$$

Where δ_x represents unit point mass at *x*, and a finite covariance, $\sum = E_f I(X_i)I(X_i)'$. The best approximating function $f_0(x) = f(x, \theta_0)$ to $f(x)$ that $f(x, \hat{\theta})$ seeks is given by $\theta_0 = T(F)$.

According to Huber [4] and Shao [9],

$$\hat{\theta} - \theta = \frac{1}{n}\sum_{i=1}^{n} I(X_i) + \frac{d}{n} + \varepsilon_n \tag{29}$$

with $\varepsilon = I(n^{-1})$ with mean $O(n^{-2})$, thus d/n is the bias of $\hat{\theta}$. The estimator can be de-bias using resampling methods (jackkniffing or bootstrapping) to remove the d/n term. Using the maximum likelihood approach,

$$I(x) = J^{-1}u_0(x) \tag{30}$$

with $J = -E_f \partial^2 \log f(X_i, \theta_0)\partial\theta\partial\theta'$.

Proposition 3.2 (Bias and Variance of the Proposed Density Estimator)**.** *Let* $f_0(x) = f(x, \theta_0)$ *and* $\theta_0 = T(F)$, *the parametric approximant to f, and* $r = f/f_0$. *The proposed semiparametric estimator has*

(a)

$$E\hat{f}(x) = f(x) + \frac{1}{24}\mu_4 h^4 f_0(x) r^{(iv)}(x) + O(h^2/n + h^4 + n^{-2}) \tag{31}$$

(b)

$$Var(\hat{f}(x)) = \frac{R(K)f(x)}{nh} - \frac{f(x)^2}{n} + O(h/n + n^{-2}). \tag{32}$$

Proof of Expectation of the Proposed Estimator. The proof relies on the fourth order Taylor expansion of

$$\frac{f(x, \hat{\theta})}{f(X_i, \hat{\theta})} \doteq \frac{f_0(x)}{f_0(X_i)} + \frac{f_0(x)}{f_0(X_i)}\{u_0(x) - u_0(X_i)\}'(\hat{\theta} - \theta). \tag{33}$$

The general form of this expansion is

$$\hat{f}(x) = f(x) + hzf'(x) + \frac{1}{2}h^2z^2 f''(x)$$

$$+ \frac{1}{9}h^3z^3 f'''(x) + \frac{1}{24}h^4z^4 f^{(iv)}(x). \tag{34}$$

Thus, expanding the above (33) in the light of the general form gives

$$\hat{f}(x) = f^*(x) + V_n(x) + \frac{1}{2}W_n(x) + \frac{1}{6}M_n(x) + \frac{1}{24}N_n(x), \tag{35}$$

where

$$f^*(x) = \overline{A}_n \tag{36}$$

$$V_n(x) = \overline{B}_n(\hat{\theta} - \theta) \tag{37}$$

$$W_n(x) = (\hat{\theta} - \theta_0)' \overline{C}_n(\hat{\theta} - \theta) \tag{38}$$

$$M_n(x) = (\hat{\theta} - \theta_0)^{*'} \overline{H}_n(\hat{\theta} - \theta_0) \tag{39}$$

$$N_n(x) = D^{*'} \overline{L}_n D^*. \tag{40}$$

Where all the representations are in terms of averages of *iid* variables;

$$A_i = K_h(X_i - x)\frac{f_0(x)}{f(X_i)}, \tag{41}$$

$$B_i = -K_h(X_i - x)\frac{f_0(x)}{f_0(X_i)}\{u_0(X_i) - u_0(x)\}, \tag{42}$$

$$C_i = K_h(X_i - x)\frac{f_0(x)}{f_0(X_i)} w(x, X_i), \tag{43}$$

$$H_i = K_h(X_i - x)\frac{f_0(x)}{f(X_i)} q(x, X_i), \tag{44}$$

$$L_1 = K_h(X_i - x)\frac{f_0(x)}{f(X_i)} p(x, X_i) \tag{45}$$

with

$$D = (\hat{\theta}_1 - \theta_{01}, ..., \hat{\theta}_n - \theta_{0p}) \tag{46}$$

$$D^* = \{(\hat{\theta}_1 - \theta_{01})^2, ..., (\hat{\theta}_p - \theta_{0n})^2\} \tag{47}$$

$$G = (u_{01}(x) - u_{01}(X_i), ..., u_{0p}(x) - u_{op}(X_i)) \tag{48}$$

$$G^* = \{(u_{01}(x) - u_{01}(X_i))^2, ..., (u_{0p}(x) - u_{op}(X_i))^2\} \tag{49}$$

$$w(x, X_i) = G'G \tag{50}$$

$$q(x, X_i) = G^{*'}G \tag{51}$$

$$p(x,\ X_i) = G^{*'}G^*. \tag{52}$$

Now, with the expected value of the proposed estimator, recall that f^* has mean $f(x) + \frac{1}{24}\mu_4 h^4 f_0(x) r^{(iv)}(x) + O(h^4)$. Using (29) and the averages representation,

$$EVn(x) = n^{-1}EB_i'I_i + n^{-1}(EB_i)' d + O(n^{-2}), \tag{53}$$

$$EW_n(x) = n^{-1}T_r(EC_i EI_i I_i') + O(n^{-2}), \tag{54}$$

$$EM_n(x) = n^{-1}T_r(H_i I_i I_i') + O(n^{-2}), \tag{55}$$

$$EN_n(x) = n^{-1}T_r(L_i I_i I_i') + O(n^{-2}) \tag{56}$$

since $I_i = I(X_i)$ has mean 0. Clearly, the fourth order Taylor series approximation used involving $(\hat{\theta}_i - \theta_{0,i})^5$ terms, is of size $O_p(n^{-2})$. Hence the bias of $\hat{f}(x)$ is

$$bias(\hat{f}(x)) = \frac{1}{24}\mu_4 h^4 f_0(x) r^{(iv)}(x) + (h^2/n)b(x) + O(h^4 + n^{-2}), \tag{57}$$

for certain $b(x)$ function.

Proof of Variance of the Proposed Estimator. The variance of $f^*(x)$ is

$$\mathrm{var}(\hat{f}(x)) = \frac{R(K)}{nh} f(x) - \frac{f(x)^2}{n}. \tag{58}$$

Using (29) and the *iid* representation,

$$Var(V_n(x)) = Var(\overline{B}_n' I_n) + O(n^{-2}) \tag{59}$$

$$= n^{-1}(EB_i)' \sum (EB_i) - \{O(h^2/n)\}^2 + O(n^{-2}) \tag{60}$$

$$= O(h^4/n + n^{-2}). \tag{61}$$

For $W_n(x)$ can be seen to have uninfluential variance $O(h^4/n^2)$. The same holds for $M_n(x)$ and $N_n(x)$ respectively. Finally,

$$Cov\{f^*(x), V_n(x)\} = n^{-1}(EB_i)' EA_i I_i + O(n^{-2}) \tag{62}$$

$$= O(h^2/n). \tag{63}$$

The combination of all these gives the necessary variance expression.

Clearly from the foregoing, the result is tractable with its simplicity; with the bias and variance being affected by only the parametric estimation noise by quite small $O(h^2/n + n^{-2})$ order. This is because $\hat{\theta}$ is close to θ_0 and $\hat{f}(x)$ estimator uses only $X_i's$ close to x, granting $u_0(X_i)$ close to $u_0(x)$. It is however, not the case considering only the correction function $\hat{r}(x)$. Consistency of the proposed density estimator is guaranteed by both $h \to 0$ (forcing bias towards 0) and $nh \to \infty$ (making variance go to 0). Additionally, if the parametric model is accurate, r function is equal to 1 and the bias is only $O(h^4 + h^2/n)$.

3.3. Normal start estimate

The normal start estimator takes the form $\hat{\sigma}^{-1}\phi(\hat{\sigma}^{-1}(x - \hat{\mu}))$, obtainable via maximum likelihood estimates $\hat{\mu} = n^{-1}\sum_{i=1}^{n} X_i$ and $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^{n}(X_i - \mu)^2$, the debias form uses $n - 1$ in the denominator. Thus, the density estimator is given by

$$\hat{f}(x) = \frac{1}{\hat{\sigma}}\phi\left(\frac{x - \hat{\mu}}{\hat{\sigma}}\right)\frac{1}{n}\sum_{i=1}^{n}\frac{K_h(X_i - x)}{\frac{1}{\hat{\sigma}}\sigma\left(\frac{X_i - \hat{\mu}}{\hat{\sigma}}\right)} \tag{64}$$

$$= \frac{1}{n}\sum_{i=1}^{n} K_h(X_i - x)\frac{\exp-\left\{\frac{1}{2}(x - \hat{\mu})^2/\hat{\sigma}^2\right\}}{\exp-\left\{\frac{1}{2}(X_i - \hat{\mu})^2/\hat{\sigma}^2\right\}}. \tag{65}$$

3.4. Mean-square error

An important concept commonly and conveniently used to measure estimation precision is the mean-square error.

$$MSE(\hat{f}(x)) = E(\hat{f}(x) - f(x))^2 \tag{66}$$

$$= Bias(\hat{f}(x))^2 + \operatorname{var}(\hat{f}(x)) \tag{67}$$

$$\approx \left(\frac{1}{24} f^{(iv)}(x)h^4\mu_4\right)^2 + \frac{f(x)R(K)}{nh} - \frac{f(x)^2}{n} \tag{68}$$

$$= \frac{\mu_4^2}{24^2} f^{(iv)}(x)^2 h^8 + \frac{f(x)R(K)}{nh} - \frac{f(x)^2}{n} \tag{69}$$

$$= AMSE(\hat{f}(x)). \tag{70}$$

It is called asymptotic mean-square-error since the approximation uses asymptotic expansions. It is a function of the sample size *n*, the bandwidth *h*, the kernel function through $(\mu_4$ and $R(K))$, and changes with *x* as $f^{(iv)}(x)$ and $f(x)$ changes.

Also, the squared bias is increasing in *h* while the variance is decreasing in *nh*. Thus, for $MSE(\hat{f}(x))$ decreases as $n \to \infty$, both bias and variance must become small. Hence, as $n \to \infty$, $h \to 0$ and $nh \to \infty$. Thus, the bandwidth must decrease at a smaller rate than the sample size. This is sufficient to establish the pointwise consistency of the estimator. Hence $\forall x$, $\hat{f}(x) \to pf(x)$ as $n \to \infty$.

3.5. Asymptotically optimum bandwidth

This refers to the value of *h* that minimizes the Asymptotically Mean Integrated Square Error (*AMISE*). This bandwidth, *h* is obtained by taking the derivative of the *AMISE* with respect to *h* and set it to 0. Thus

$$\frac{d}{dh} AMISE = \frac{d}{dh}\left(\frac{\mu_4}{24^2} f^{(iv)}(x)h^8 + \frac{f(x)R(K)}{nh} - \frac{f(x)^2}{n}\right) \tag{71}$$

$$= \frac{8\mu_4^2}{24^2} f^{(iv)}(x)^2 h^7 - \frac{f(x)R(K)}{nh^2} \tag{72}$$

$$0 = \frac{8\mu_4^2}{24^2} f^{(iv)}(x)^2 h^7 - \frac{f(x)R(K)}{nh^2} \tag{73}$$

$$h^9 = \frac{24^2 f(x)R(K)}{8\mu_4^2 R(f^{(iv)}(x))n} \tag{74}$$

$$\Rightarrow h_0 = \left(\frac{24^2 f(x)R(K)}{8\mu_4^2 R(f^{(iv)}(x))}\right)^{\frac{1}{9}} n^{-\frac{1}{9}} \tag{75}$$

$$\therefore h_0 = \left(\frac{24^2 \hat{f}(x)\hat{R}(K)}{8\mu_4^2 \hat{R}(\hat{f}^{(iv)}(x))}\right)^{\frac{1}{9}} n^{-\frac{1}{9}}. \tag{76}$$

The optimal bandwidth has order $O(n^{-\frac{1}{9}})$. For higher order kernels, the convergence rate is slower implying that larger bandwidths than that used for second order kernels are permissible. This is due to the fact that higher order kernels have smaller bias, hence can allow larger bandwidths. By substituting h_0 into (69), the variance and bias terms yields an order of $O(n^{-\frac{8}{9}})$. It is worth noting that this convergence rate is fast approaching the parametric rate of convergence and hence the better the density estimator.

4. Simulation Results

Simulation studies performed on the proposed estimator revealed that the estimator has an asymptotic mean square error of 0.9999542 which is almost unity. This makes it more tractable for use in practical applications. The challenge of its use maybe based on the bandwidth estimation owing to the fact that, numerical approximations are used in estimating the optimum bandwidth with its attendant approximation errors. This problem can be addressed by using a *possible best search* where the analytical solution seems unsatisfactory since a maximum deviation of between 10–15% of the optimum bandwidth often produce satisfactory results (Jones et al. [6]; Scott [8]). The comparative performance of the proposed estimator against its ordinary kernel counterpart showed that the estimator is superior using the first five Marron-Wand densities. The results are tabulated below.

MISE values for Proposed estimator, $\hat{f}_p$ and Kernel functions		
Marron-Wand Density	$\hat{f}_p$	Kernel
Gaussian	58.18089	68.47813
Skewed	112.5738	129.0788
Strongly Skewed	420.6225	450.8423
Kurtotic	558.4034	589.873
Outlier	6320.694	6459.58

From the plots above, it can be seen that the optimum bandwidth of the estimator is around .30 when the data driven default is actually 0.2338 underscoring its ability to admit higher bandwidths hence increasing precision of estimation.

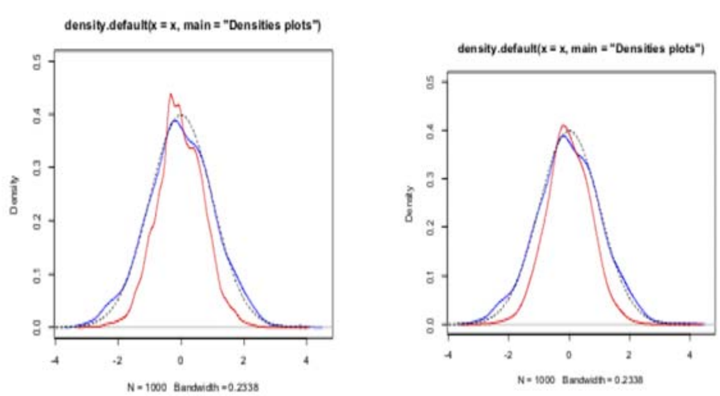

Figure 1. Comparative plots for the proposed density and its competitors and bandwidths (a) $h = 0.1$ (b) $h = 0.2$.

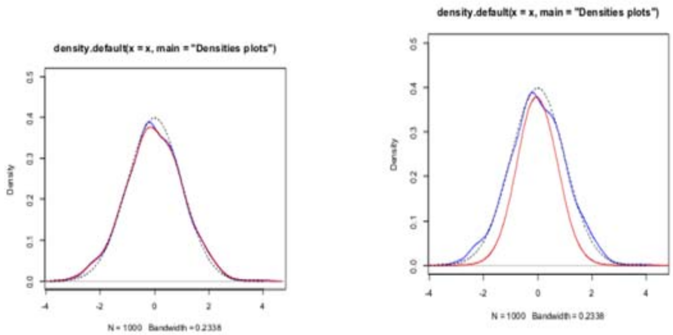

Figure 2. Comparative plots for the proposed density and its competitors and bandwidths $h = 0.3$ and $h = 0.4$ respectively.

References

[1] I. S. Abramson, On bandwidth variation in kernel estimates-a square root law, The Annals of Statistics (1982), 1217-1223.

[2] A. C. Cameron and P. K. Trivedi, Microeconometrics Methods and Applications, Cambridge University Press (2005).

[3] N. L. Hjort and I. K. Glad, Nonparametric density estimation with a parametric start, Annals of Statistics 23(3) (1995), 882-904.

[4] P. J. Huber, Robust statistics, Wiley, New York, 1981.

[5] M. Jones, Variable kernel density estimates and variable kernel density estimates, Australian Journal of Statistics 32(3) (1990), 361-371.

[6] M. C. Jones, O. Linton and J. P. Nielsen, A simple bias reduction method for density estimation, Biometrika 82(2) (1995), 327-38.

[7] D. Ruppert and D. B. Cline, Bias reduction in kernel density estimation by smoothed empirical transformations, The Annals of Statistics (1994), 185-210.

[8] D. W. Scott, Multivariate density estimation: theory, practice, and visualization, John Wiley & Sons, 2015.

[9] J. Shao, Second-order differentiability and jackknife, Statistica Sinica 1 (1991), 185-202.

[10] B. W. Silverman, Density Estimation for Statistics and Data Analysis, Chapman and Hall, 1986.

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