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The Possibility of Angle Trisection (A Compass-Straightedge Construction) Kimuya M Alex

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Abstract: The objective of this paper is to provide a provable solution of the ancient Greek problem of trisecting an arbitrary angle employing only compass and straightedge (ruler). (Pierre Laurent Wantzel, 1837) obscurely presented a proof based on ideas from Galois field showing that, the solution of angle trisection corresponds to solution of the cubic equation; $x^3 - 3x - 1 = 0$, which is geometrically irreducible [1]. The focus of this work is to show the possibility to solve the trisection of an angle by correcting some flawed methods meant for general construction of angles, and exemplify why the stated trisection impossible proof is not geometrically valid. The revealed proof is based on a concept from the Archimedes proposition of straightedge construction [2, 3].

Key Words: Angle trisection; Compass; Ruler (Straightedge); Classical Construction; GeoGebra Software; Greek's geometry; Cubic equation; Plane geometry; Solid geometry

Notations

$\angle$	Denotes an angle
$\overline{\cap\forall}$	Denotes a straight line and a length
$2D$	Two Dimensional
$3D$	Three Dimensional

1. Introduction

The early Greek mathematicians were unable to solve three problems of compass-ruler (straightedge) construction; the 'trisection of an angle', 'how to double the volume of a given cube', and the problem of 'squaring a circle'. Eventually the problems were assumed to be unsolvable under the restrictions imposed by the Greek mathematicians. The scope of this paper is restricted on the trisection of an arbitrary angle. The unclear proof of angle trisection impossibility was established, based on ideas from Galois field of algebra, and it stated that 'The trisection of an angle corresponds to the solution of a certain cubic equation whose solution cannot be sought under the Greek's rules of Geometry' [1, 2]. Three algebraic constraints exists which state; a length can only be constructible if and only if it represents a constructible number, an angle is constructible if and only if its cosine is a constructible number, and, a number is constructible if and only if it can be written in the four basic arithmetic operations and the extraction of the square roots but not on higher roots [4]. These three conditions are quite fashionable in Euclidean constructions, and they enabled the early mathematicians correctly justify the construction of all the angles multiples and sub multiples of 15 and the associated regular polygons. However, this novel paper presents a correct algorithm justifying that the stated algebraic specifications does not apply to all the plane geometrical problems, by drawing out some limitations in the presented proof of impossibility. For instance, some angles such as $45°$, $90°$ and $180°$ are trisectible following the Euclidean rigor of construction, but there trisection methods could not be adopted since they do not generalize for the partitioning of all the angles into the desired ratio [5]. Moreover, it is geometrically possible to bisect a line

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segment between two points in a plane, as well as the partitioning a line segment into equal fractions using the Greek's tools of geometry [6, 7]. The presented proof of the impossibility does not show that some of these cases are constructible. Considering the trisection of and angle as a cubic equation translated the problem from a $2D$ (plane geometry) problem as it should be sought, to a $3D$ (solid geometry) problem, which involve equations of the form n^3 as discussed in section 1.2 . This was a serious misconstruction and due to inability to geometrically solve the cubic equations, and the fact that no geometrical algorithm has been presented to solve the partitioning of an angle into a given ratio, mathematicians wicked to pseudo mathematical approaches which do not redress the problem with the desired degree of correctness [9]. This paper relies on a simple concept of the Archimedes theorem of straightedge which stated "If we are in possession of a straightedge that is notched in two places, then it is possible to trisect an arbitrary angle", by revealing a geometrical solution for this ancient problem, contrally to the Archimedes approach of using a marked straightedge. An elementary proof of solving the posed problem of angle trisection is then exposed from the construction, by trisection of $48°$ and $60°$ angles.

1.1 To show that any three points not lying on a straight line lie in the same plane

When two rays in a plane share a common endpoint, an angle is produced between the two rays. An example is in a geometrical figure such as triangle. Any two sides of a triangle have a common point at the vertices and thus angles of some size are defined between the two sides of the figure. In this section, a brief discussion about any three points, one not collinear with the other two in a plane is presented. Consider the following theorem:

Theorem 1*:* Any three points not lying in a straight line lie only in same plane, and every triangle lies only in one plane [8].

Considering three points; A, B, and C, not lying in one straight line but all connected together by straight lines as shown on figure (1), it is not possible for all the three points to lie on different planes. To justify this proposition, let points A, B, and C lie on two distinct planes; M and N. Since both point A and B lie on plane M, the straight line AB lie on plane M. Also, since points A and B lie on plane N, the straight line AB also lie on plane N. Therefore, plane M and plane N have line AB in common (they intersect along line AB). Point C does not lie in line with AB and therefore not common for both planes. Thus it is not possible for points A, B, C to lie in both planes M and N. Considering triangle ABC, the whole triangle lies only in one plane. Hence, for any plane containing all of triangle ABC must also contain its three vertices A, B, and C. This shows that the whole of any triangle lies only in one plane. Considering $\angle ABC$ as the acute angle to be trisected, it is required that the points defining the trisection lines lie in the curve subtending $\angle ABC$ at point B, and that point B is defined in a two dimensional coordinates system. Therefore, the problem of trisecting an angle has to be sought following the classical rules of Euclidean geometry.

1.2 The Mistake In Pierre Laurent Wantzel's Proof Of Angle Trisection Impossibility (1837)

Theorem 1 illustrates how three points defining a given angle, and not collinear lie in the same plane, and that in plane geometry two different planes cannot share all points in common.It is pellucid that algebra

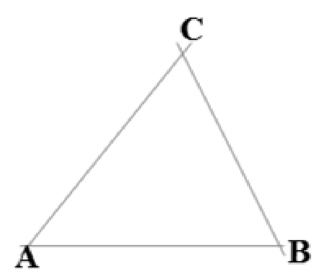

Fig. 1 Illustration of a plane geometry figure.

has well been applied in justification of plane geometric problems and the underlying concepts correctly verified. In this sense, degree two polynomials have been employed in defining constructions in a plane such as; determining the quadratic equations defining some angle bisection lines. The scope of this paper is restricted in Euclidean plane geometry of straightedge and compass construction. Plane geometric construction is governed by many propositions and theorems, which greatly influenced the development of geometry. As stated earlier, some few angles are geometrically trisectible, it is possible to bisect both an angle and a straight line, as well as dividing a straight line segment into the desired number of equal portions. It is also possible to construct lines of magnitude $\sqrt{2}$ and $\sqrt{5}$. The existence of these well justified constructions implies the great probability of trisecting a given angle, or the construction of an angle of a certain ratio. The third algebraic condition specified in section 1.0 limits plane geometric constructions from advancing in higher orders of roots, above the square roots of numbers. According to this article, this condition is not fashionable in restricting the extraction of higher order roots above square roots, in geometry, based on the fact that a line segment can be geometrically fractioned into the required ratio [8]. Depending on the application, it is quite difficult to determine the magnitudes of the sliced portions of a line segment, and algebraically, this implies the possibility of constructing numbers which do not represent constructible numbers, a case contradicting the stated algebraic condition. This section in a simple and brief approach discusses the angle trisection impossibility, to reveal how the algebraic approach turned the problem from plane geometry problem into a solid geometry problem and thus the impossibility. The impossibility proof was centered on concepts from the Galois field. Therefore, based on the ancient proof, the problem can be stated as: Define a configuration to be a finite collection C of points, lines, and circles in the Euclidean plane. Define a construction step to be one of the operations to enlarge the collection C as: Given two distinct points A and B in C, join points A and B, using a straight line and add $\overline{AB}$ to C. Given a third point O in C, construct a circle with center O and radius $\overline{AB}$ of the line segment joining A and B, and add it to C using a compass. Given two different curves γ and u in C (such that γ and u are either a line or a circle in C), select a point T that is common to both γ and u and add it to C. Therefore a point, line, or circle is said to be from a configuration C if it can be obtained from C after applying a finite number of construction steps. From these deductions, there are only two positions in which the point T can be located. The point T however, quadrasects angle AOB and it must be outside the initial plane defined by radius $\overline{AB}$. Trisection of an angle may not be easily solved in such a linear construction, but a deeper look at the problem would obligate some serious exploration of the relationship between some chords and curves in a circular plane, as presented in [3]. The wantzel's proof considered the lemma; there is no power of 2 that is evenly divisible by 3 which was employed to demonstrate the angle trisection impossibility using concepts from Galois field of numbers. A proof upon this lemma can be described as follows:

Corollary: Let Q be a field, and let R be an extension of Q that is constructible out of R by a finite order of quadratic extensions. Then Q does not contain any cubic extensions $V$of R.

Proof: If Q contained a cubic extension $V$of R, then the dimension of Q over R would be a multiple of three. On the other hand, if Q is obtained from R by a tower of quadratic extensions, then the dimension of Q over R is a power of two. It can therefore be stated, any point, line, or circle that can be constructed from a configuration C is definable in a field obtained from the coefficients of all the objects in C after taking a finite number of quadratic extensions, whereas trisection of an angle such as $\angle ABC$ will basically be definable in a cubic extension of the field generated by the coordinates of A, B, C. Based on these coordinates, three plane angles $\angle ABC$, $\angle BAC$ and $\angle ACB$ can be constructed to represent three different planes such that; the three plane angles add up to less than four right angles and any two of them add up to more than the third one [6, 8]. These conditions can be met in a solid geometric construction, when for example, the three plane angles made from isosceles triangles of equal legs meet their vertices at a common endpoint. In analytic Euclidean plane geometry, an angle is genetically defined in a two dimensions (x, y- coordinates) with x and y as real numbers, and not in three dimensions as presented in the Pierre Wantzel's cubic equation of the impossibility. From this discussion, it is evident that the presented proof of impossibility dictates rotation of objects to reach some accuracy as applied in solutions using 'other methods' of trisecting an angle [24], and these operations are prohibited in Euclidean plane geometry. Therefore, the impossibility proof that an angle cannot be divided into a certain fraction, or that other angles not under angles of base $60°$ cannot be constructed has no geometric precision. Thus the trisection of an angle is typically a plane geometry problem (2*D*), and not a 3*D* problem as it has been sought.

2. Hypothesis

In a typical plane geometric construction, the relation between two angle can be defined by ratios; $a/b = x$, with a and b as the curves subtending the larger angle to the smaller angle at a point respectively. Therefore, considering two angles, α and β such that $\alpha/\beta = \delta$, then, $\delta \cong x$, where a and b correspond to α and β respectively. Thus taking the ratios between any two given angles; $\alpha/\beta = \delta$, some cases would yield a ratio from which to derive the relation $a/b = x$ is geometrically difficult. Therefore, for consistency, it is important to choose a constant difference between two angles, as such the difference between the constructible angles multiples of 15 is $15°$, and multiple or sub multiples of $15°$. This paper therefore considered the relation between any two angles at a difference of $10°$, or a difference multiple of 10 from each other in there descending order. The most significant ratio was $60°/20° = 3:1$. This consideration is due to the fact that the $60°$ angle form the base for the construction of all the angles multiples of 15 [3]. The novelty of these ratios rose due to the need to generate a method in which a particular considerable distance between two points transform equally into two different points without rotation or sliding of objects to locate the new points. Consider figure (2). The resolution of this work is to geometrically ratify that the conferred relations exists, such that: $\overline{AB} = \overline{AD'} = \overline{D'E} = \overline{AC} = r$, where r is the radius of the circle, and that $\angle BAD = \angle EAD' = \angle AED'$. This implies $3\angle BAD = \angle CAB$.

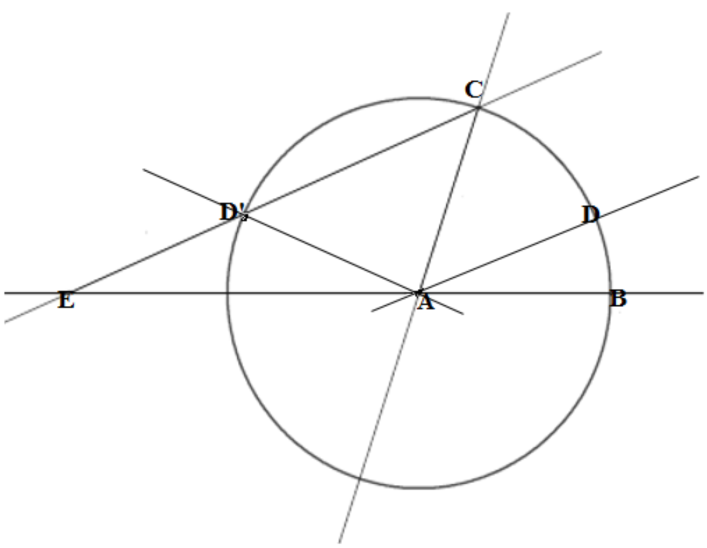

Fig. 2 Transformation of point D to points D' and E geometrically.

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3. Materials and Methods

3.1 Materials

The required mathematical tools in solving this problem include;

- Compass
- Ruler (straightedge)
- Piece of a drawing paper
- Pencil
- Computer
- GeoGebra Software installed in the computer.

3.2 Methodology

This section has improved on a construction presented for the classical construction of angles in general [3]. The present theorem begins by depicting how one can construct angles multiples of 2 by constructing a $32°$ angle using the usual Greek's tools of geometry, and then the construction of angles multiples of 5, and 10 or their sub multiples, by constructing an angle of $40°$, and the methods are justified by trisecting angles of $48°$ and $60°$ respectively.

3.2.1 Construction of $\mathbf{32}°$ angle using a ruler (straightedge) and compass only.

(1) Draw a straight line between two points A and B.

(2) Mark a point C equidistant $\overline{AB}$ from both A and B and draw an arc BC centered at A.

(3) Join the chord $\overline{CB}$, and construct its bisection through a point D to cut curve BC at E.

(4) Construct the bisection of $\angle EDB$ to cut curve BE at F.

(5) Further, construct the bisection of $\angle EDF$ to cut curve FE at G.

(6) Position the pair compass at point E and make an arc of length $\overline{DF}$ to cut curve EC at H.

(7) With the compass at point H, mark point I of distance $\overline{DG}$ between carve HE. $\angle IAB = 32°$.

3.2.2 Trisection of $\angle\mathbf{48}°$ Using Compass-ruler (Straightedge) Construction to Proof the Construction of $\angle\mathbf{32}°$

Figure (4) depicts the results obtained after carrying out the following steps of construction using a ruler and compass.

(1) Draw a straight line between two points A and B.

(2) Mark a point C equidistant $\overline{AB}$ from both A and B and draw an arc BC centered at A.

(3) Join the chord $\overline{CB}$, and construct its bisection through a point D to cut curve BC at E.

(4) Construct the bisection of $\angle EDB$ to cut curve BE at F.

(5) Further, construct the bisection of $\angle EDF$ to cut curve FE at G.

(6) Position the compass at point E and make an arc of length $\overline{DF}$ to cut curve EC at H.

(7) With the compass at point H, mark point I of distance $\overline{DG}$ between carve HE.

(8) Construct the bisector of $\angle IAB$ to cut curve BI at point J.

(9) With the compass at point I, mark an arc of radius BJ to cut curve IC at K.$\angle KAB = 48°$.

(10) Reflect point B about A to produce point B'.

(11) Again using chord BJ, place the compass at B' and mark a point J' on the arc $B'C$. J' is an image of point J.

(12) Draw a straight line from point K through J' to meet the produced diameter BB' externally at a point M.

3.2.3 Proof

Consider figure (4).

3.2.3.1 To show $\overline{\boldsymbol{MJ'}} = \overline{\boldsymbol{AJ'}} = \overline{\boldsymbol{AB}}$

This part of proof employs the compass equivalence theorem to justify that; length MJ' is equal to the radius of the original (blue) circle, by drawing a circle of radius $\overline{MJ'}$, and center J'. If $\overline{MJ'} = \overline{AJ'}$, then the circumference of the circle with radius $\overline{MJ'}$ has to pass through point A as shown in figure (5).

Thus, the circle in blue circumference is congruent to the circle defined by the circumference in orange, implying: $\overline{MJ'} = \overline{AJ'} = \overline{AB}$, circles radii. Therefore, this proof indicates that if a trisection is not precisely correct, the two circles would not be congruent.

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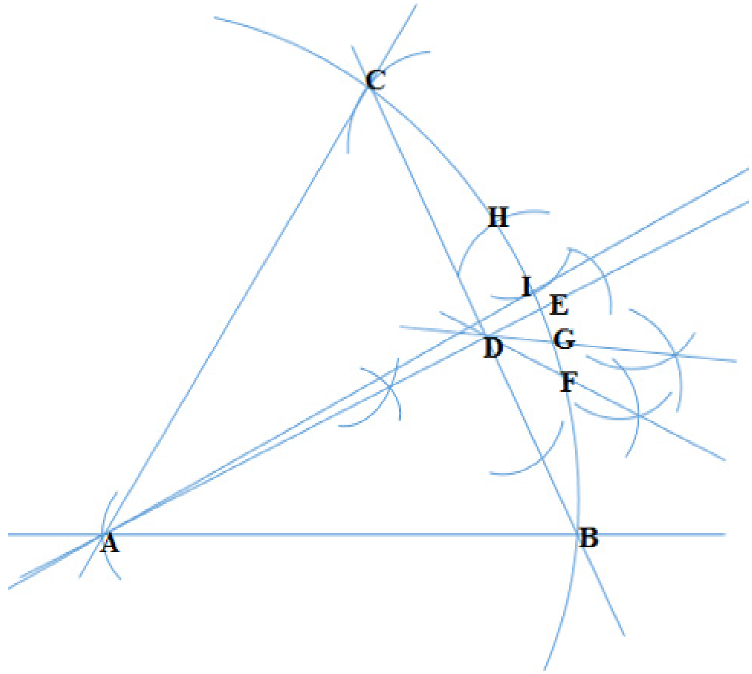

Fig. 3 Construction of 32° angle.

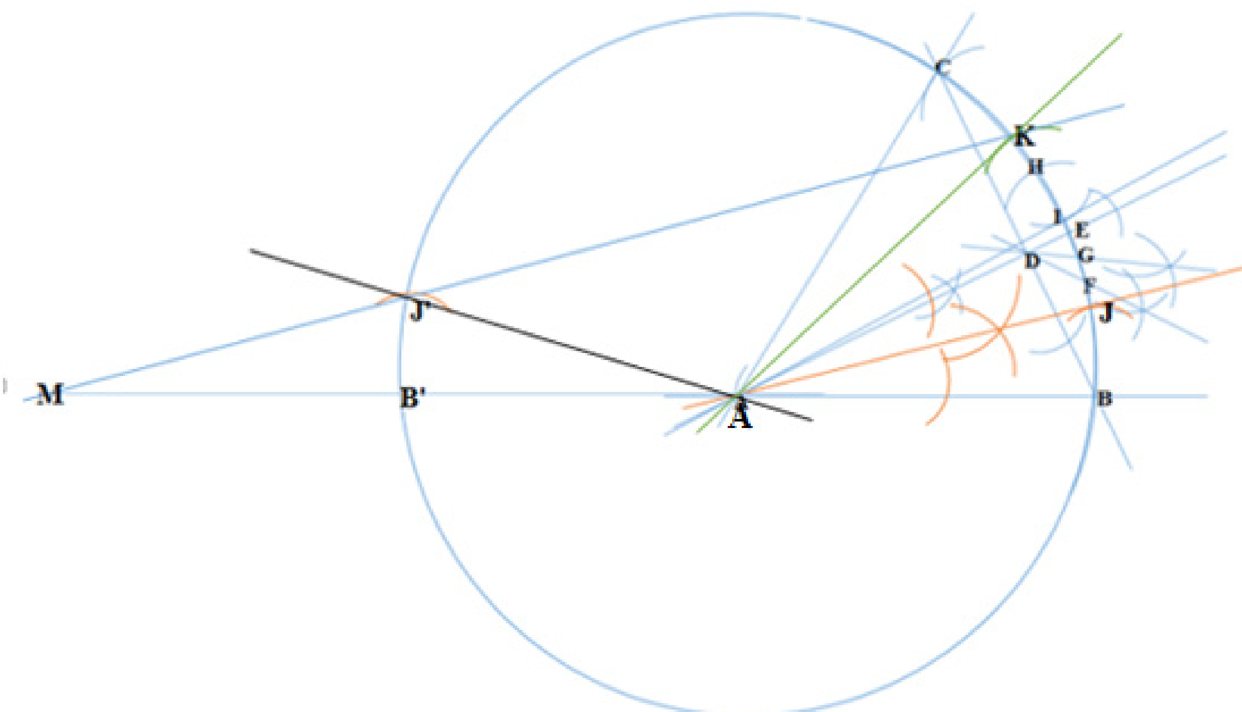

Fig. 4 Geometrical Proof of Trisecting 48° Angle.

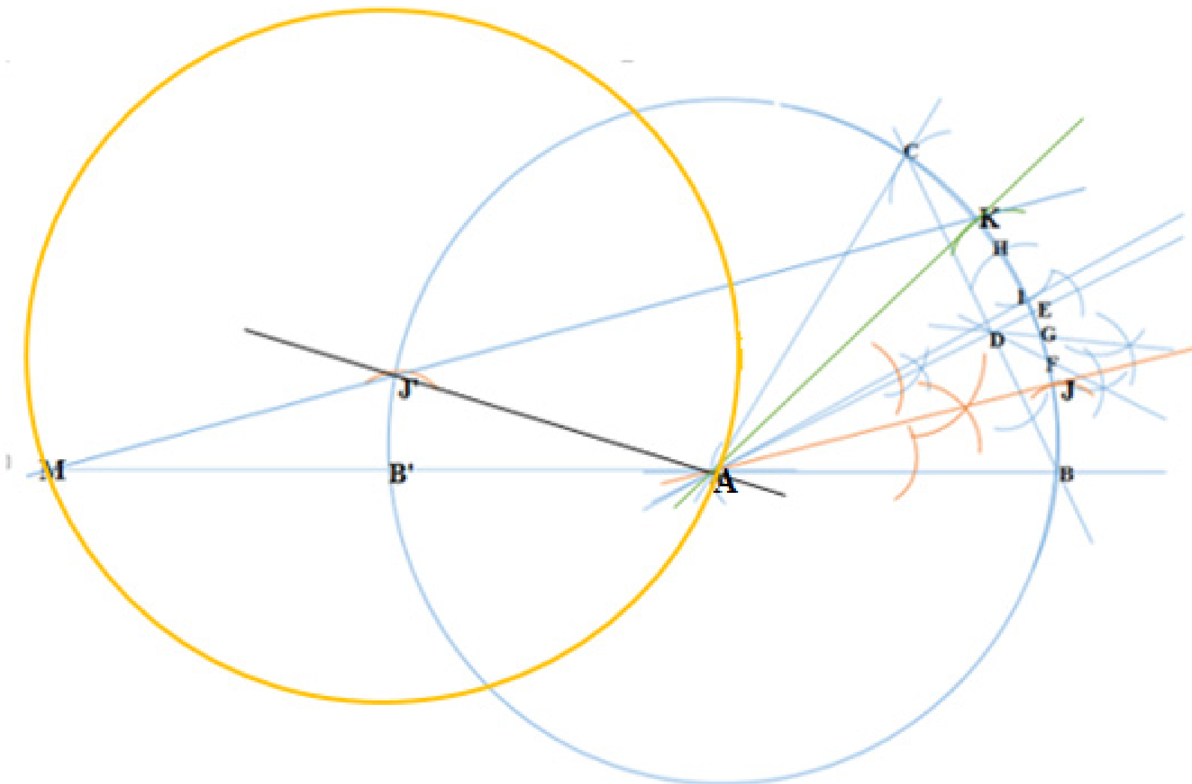

Fig. 5 To Justify that $\overline{MJ'} = \overline{AJ'} = \overline{AB}$.

3.2.3.2 To show that $\angle \boldsymbol{KAB} = \boldsymbol{3}\angle \boldsymbol{JAB}$

From figure (4), $\overline{AB} = \overline{BC} = \overline{CA}$, implying triangle ABC is equilateral. $\angle EAB = 30°$. The goal of this section of proof is to show that $\angle IAB = 32°$. Applying a concept of transformations, the point J is reflected on the circumference of the circle, on the plane BCB' to yield point J' as shown above.

Let $\angle JAB = \emptyset$.

Triangle $B'AJ' \equiv BAJ$, by the property SAS. It follows that, $\angle B'AJ" = \emptyset$. (1)

Since: $\overline{MJ'} = \overline{AJ'}$, $\angle J'MB' = \angle B'AJ' = \emptyset$ (Base angles of isosceles triangle $AJ'M$).

Again, $\angle AJ'K = 2\emptyset$ (Sum of two interior angles is equal to the size of the opposite exterior angle of the triangle). We also have; $\angle AJ'K = \angle J'KA = 2\emptyset$, (Base angles of isosceles triangle $J'AK$).

From these deductions, $\angle J'AK = 180° - 4\emptyset$. (2)

Consider; $\angle B'AJ' + \angle J'AK + \angle KAB = 180°$ (Angles on a straight line). Making $\angle KAB$ the subject; $\angle KAB = 180° - (\angle B'AJ' + \angle J'AK)$. (3)

Substituting equations (1) and (2) in equation (3);

$$\angle KAB = 180° - (\emptyset + 180° - 4\emptyset)$$

$\angle KAB = -\emptyset + 4\emptyset = 3\emptyset$. (4)

Thus from (4),$\angle KAB = 3\angle JAB$.

Since $\angle KAB = 48°$, $\angle JAB = 16°$ (a bisector of $\angle IAB$), implying $\angle IAB = 32°$.

Figure (6) demonstrates the presented proof of trisecting $\angle 48°$ using the GeoGebra software.

3.3 Construction of $\mathbf{40°}$ angle using compass and ruler (straightedge) only.

Carrying out the following steps of a construction would help to construct an angle of $40°$ using only a compass and a ruler, as illustrated in figure (7).

(1) Draw a straight line between two points A and B.

(2) Mark a point C equidistant $\overline{AB}$ from both A and B and draw an arc BC centered at A.

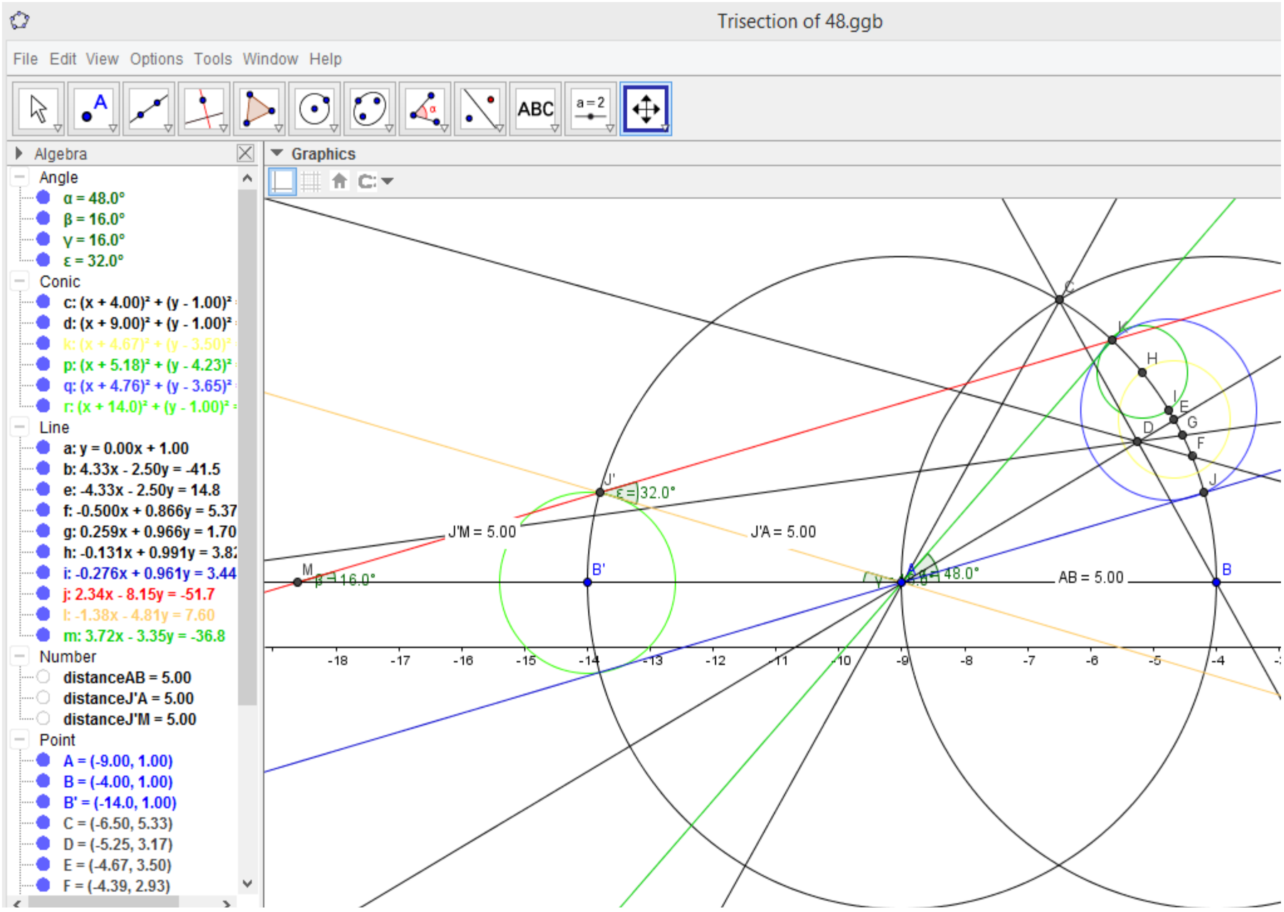

Fig. 6 To illustrate the Trisection of $48°$ angle.

(3) Join the chord $\overline{CB}$, and construct its bisection through a point D to cut curve BC at E.

(4) Construct the bisection of $\angle EDB$ to cut curve BE at F.

(5) Further, construct the bisection of $\angle EDF$ to cut curve FE at G.

(6) Position the pair compass at point E and make an arc of length $\overline{DF}$ to cut curve EC at H.

(7) With the compass at point H, mark point I of distance $\overline{DG}$ between carve HE. $\angle IAB = 32$.

(8) Construct the bisection of angle IAB to cut curve BI at J.

(9) Bisect $\angle JAB$ at K on curve BJ.

(10) Placing the compass at I, mark a point L, distance $\overline{BK}$ on curve IH. $\angle LAB = 40°$.

3.3.1 Trisection of $\mathbf{60°}$ angle to proof the construction of $\mathbf{40°}$ angle

This section presents an illustrative construction of trisecting an angle of $60°$ using GeoGebra. Consider figure (8) generated after performing the following steps of construction using the geometry software:

(1) Draw a straight line between two points A and B.

(2) Mark a point C equidistant $\overline{AB}$ from both A and B and draw an arc BC centered at A.

(3) Join the chord $\overline{CB}$, and construct its bisection through a point D to cut curve BC at E.

(4) Construct the bisection of $\angle EDB$ to cut curve BE at F.

(5) Further, construct the bisection of $\angle EDF$ to cut curve FE at G.

(6) Position the pair compass at point E and make an arc of length $\overline{DF}$ to cut curve EC at H.

(7) With the compass at point H, mark point I of distance $\overline{DG}$ between carve HE.

(8) Construct the bisection of $\angle IAB$ to cut curve BI at J.

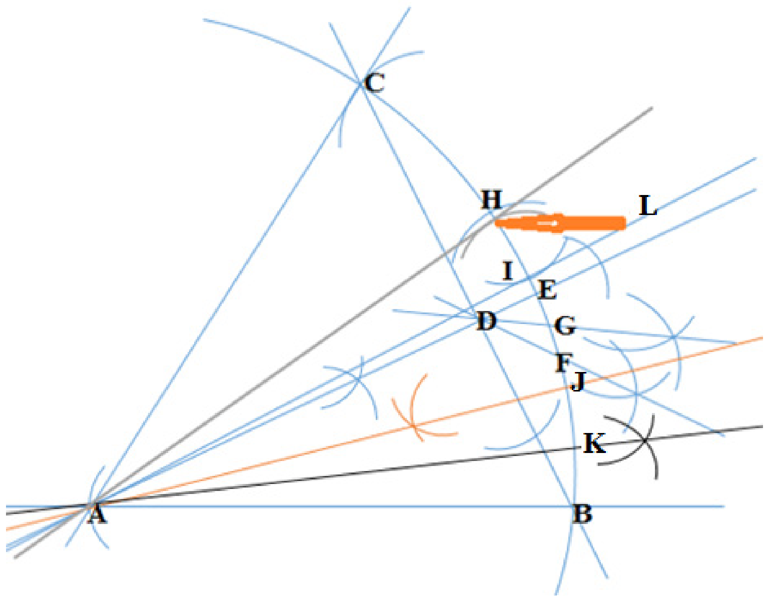

Fig. 7 Construction of $40°$ angle.

(9) Bisect $\angle JAB$ at K on curve BJ.

(10) Placing the compass at I, mark a point L, distance $\overline{BK}$ on curve IH.

(11) Construct the bisection of $\angle LAB$ to cut curve BE at M.

From the algebra window, by default the software awards the symbols α, β, γ for $\angle CAL$, $\angle LAM$, and $\angle MAB$ respectively. It is clearly shown that, $\angle CAL = LAM = \angle MAB = 20°$.

3.3.2 A Geometrical Proof that it is Possible to Trisect $\mathbf{60°}$ Angle

Consider the following steps of construction and the produced diagram;

(1) Draw a circle with radius $\overline{AB}$, and its dimeter extended outside as depicted below.

(2) Mark a point C on the circumference equidistant $\overline{AB}$ from both A and B.

(3) Join the chord CB, and construct its bisection at a point D to cut curve BC at E.

(4) Construct the bisection of $\angle EDB$ to cut curve BE at F.

(5) Further, construct the bisection of $\angle EDF$ to cut curve FE at G.

(6) Position the pair compass at point E and make an arc of radius $\overline{DF}$ to cut curve EC at H.

(7) With the compass at point H, mark point I of distance $\overline{DG}$ between carve HE.

(8) Construct the bisection of $\angle IAB$ to cut curve BI at J.

(9) Bisect $\angle JAB$ at K on curve BJ.

(10) Placing the compass at I, mark a point L, distance $\overline{BK}$ on curve IH.

(11) Construct the bisection of $\angle LAB$ to cut curve BE at M.

(12) Reflect point B about point A to get point B' as shown below.

(13) With the compass at B', mark a point M'such that, $\overline{B'M'} = \overline{BM}$. Draw a straight line from point C through M' to meet the extended diameter BB'externally at a point N.

3.3.3 *Theorem 2:* Based on the classical Greek's rules of construction it can be deduced that: "*If the terminal point of the curve or the chord subtending the "trisection angle" at the center of the circle is reflected on the opposite side at circumference in the same plane, and a line drawn from the terminal of the "trisected angle" through the mirrored point to intersect the diameter outside the circle , such that,*

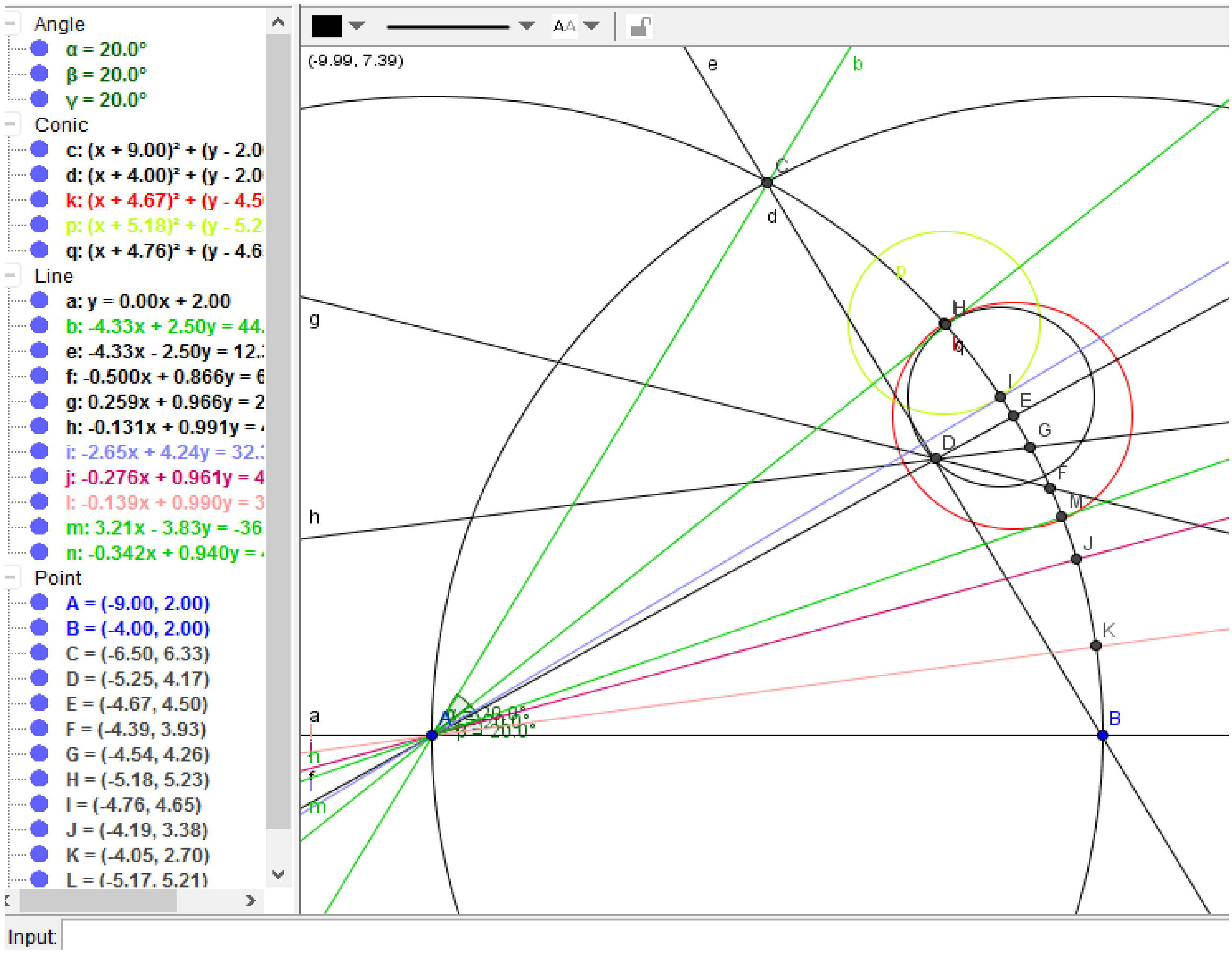

Fig. 8 **Trisection of $60°$ angle.**

the distances between the point of intersection and the reflected point is equal to the radius of the circle, then it is geometrically possible to trisect a given angle.".

A proof elaborating this proposition is presented using the following diagram. The point N is the intersection between the diameter of the circle and a line from the terminal of the arc (point C) of the acute $\angle CAB$, through M', where M' is a reflection of point M.

Let the arbitrary angle to be trisected be the acute $\angle CAB$. From the construction steps it is clear that triangle ABC is equilateral since $\overline{AB} = \overline{BC} = \overline{CA}$. Therefore this proof is to show the trisection of a $60°$ angle. Consider, $\overline{NM'} = \overline{M'A} = \overline{AC}$ (radii). $\overline{M'A}$ is a reflection of $\overline{AM}$ about point A. Thus triangle $MAB \equiv M'AB'$, by property SAS. It implies that triangles $NM'A$ and $M'AC$ are both isosceles. Applying the approach used in the previous proof and letting $\angle MAB = \theta$, then $\angle M'NA = \angle M'AN = \theta$. Thus $\angle AM'C = 2\theta$ ('sum of two interior angles in a triangle equal the size of the opposite exterior angle of the triangle'). It follows that $\angle M'CA = \angle AM'C = 2\theta$ (base angles of an isosceles triangle). Again, $\angle CAM' = 180° - 4\theta$, so that we have:

$$\angle NAM' + \angle CAM' + \angle CAB = 180°$$

(Angles in a straight line). (5)

By substituting for $\angle CAM' = 180° - 4\theta$ we have;

$$\angle CAB = 180° - \angle NAM' + \angle CAM'$$

$$\angle CAB = 180° - (\angle NAM' + 180° - 4\theta) \quad (6)$$

$$\angle CAB = -\theta + 4\theta = 3\theta$$

Therefore,

$$\angle CAB = 3\theta. \quad (7)$$

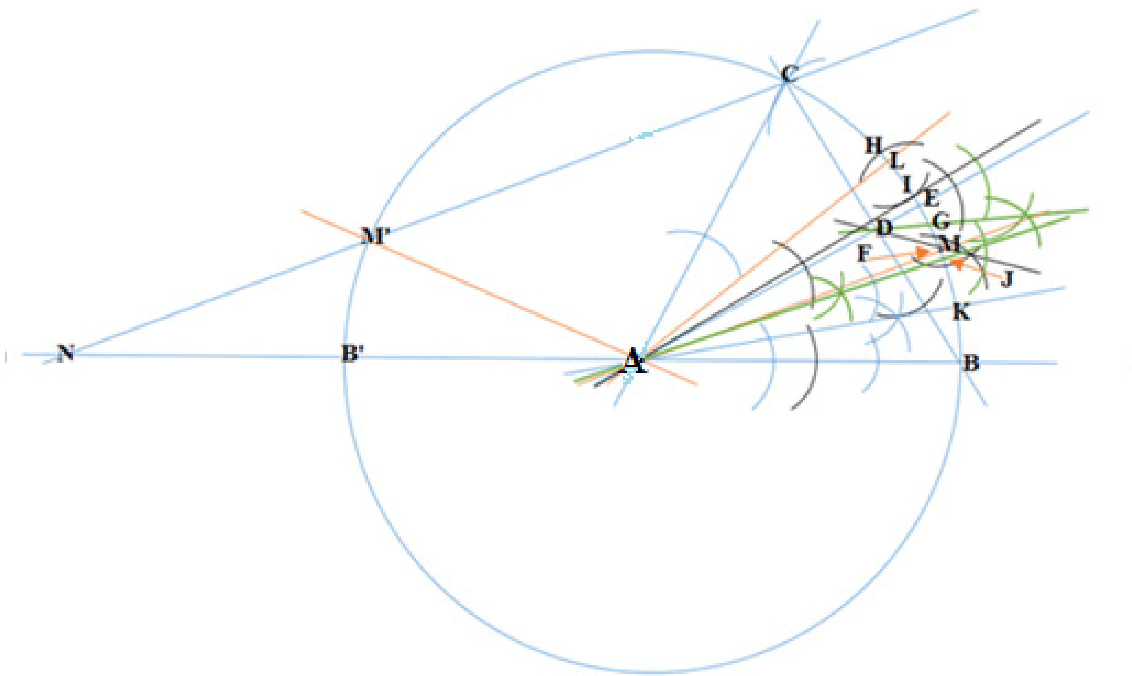

Fig. 9 Angle Trisection Proof.

Equation (7) implies that, $\angle CAB = 3\angle M'NA = 3\angle MAB \Longrightarrow {}^{1}/_{3}\angle CAB = \angle MAB$, as required.

3.3.4 Use of GeoGebra software to justify the Proof

The use of GeoGebra in this section is to exemplify two important aspects from the provided proof. First is to show line $\overline{NM'}$ lie on $\overline{NC}$ such that $\overline{NM'} = \overline{M'A}$and line $\overline{M'A}$ is a radius of the circle as discussed earlier. The other aspect verified is that; $\angle M'CA = \angle AM'C = 2\theta$.

From figure (10), it can be visualized that the presented cases $\overline{NM'} = \overline{M'A} = \overline{B'A} = 5units$ (radii), and $\angle M'NA = \angle M'AN = \theta$ where $\theta = 20°$ are absolutely correct. The methodology made it possible to trisect the $60°$, and this implies the possibility to construct all angles multiples and sub-multiples of 10.

4. Application

In this section, a rationalization of the presented proofs is made by construction of some regular polygons; (Pentagon and Nonagon), to represent some of the regular polygons which could not be geometrically solved.

4.1 Construction of a Pentagon to justify the construction of a $\mathbf{8}°$ angle using only a ruler and compass

(1) Draw a circumference of radius $\overline{AB}$.

(2) Mark a point C on the circumference, equidistant $\overline{AB}$ from both A and B centered at A.

(3) Join the chord $\overline{CB}$, and construct its bisection through a point D to cut curve BC at E.

(4) Construct the bisection of $\angle EDB$ to cut curve BE at F.

(5) Further, construct the bisection of $\angle EDF$ to cut curve FE at G.

(6) Position the pair compass at point E and make an arc of length $\overline{DF}$ to cut curve EC at H.

(7) With the compass at point H, mark point I of distance $\overline{DG}$ between carve HE.

(8) Construct the bisection of $\angle IAB$ to cut curve BI at J.

(9) Bisect $\angle JAB$ at K on curve BJ.

(10) Placing the compass at point I, make an arc of length $\overline{BK}$ to cut curve IB at L.

(11) With the compass at L, mark a point M on the circumference using chord BI as shown below.

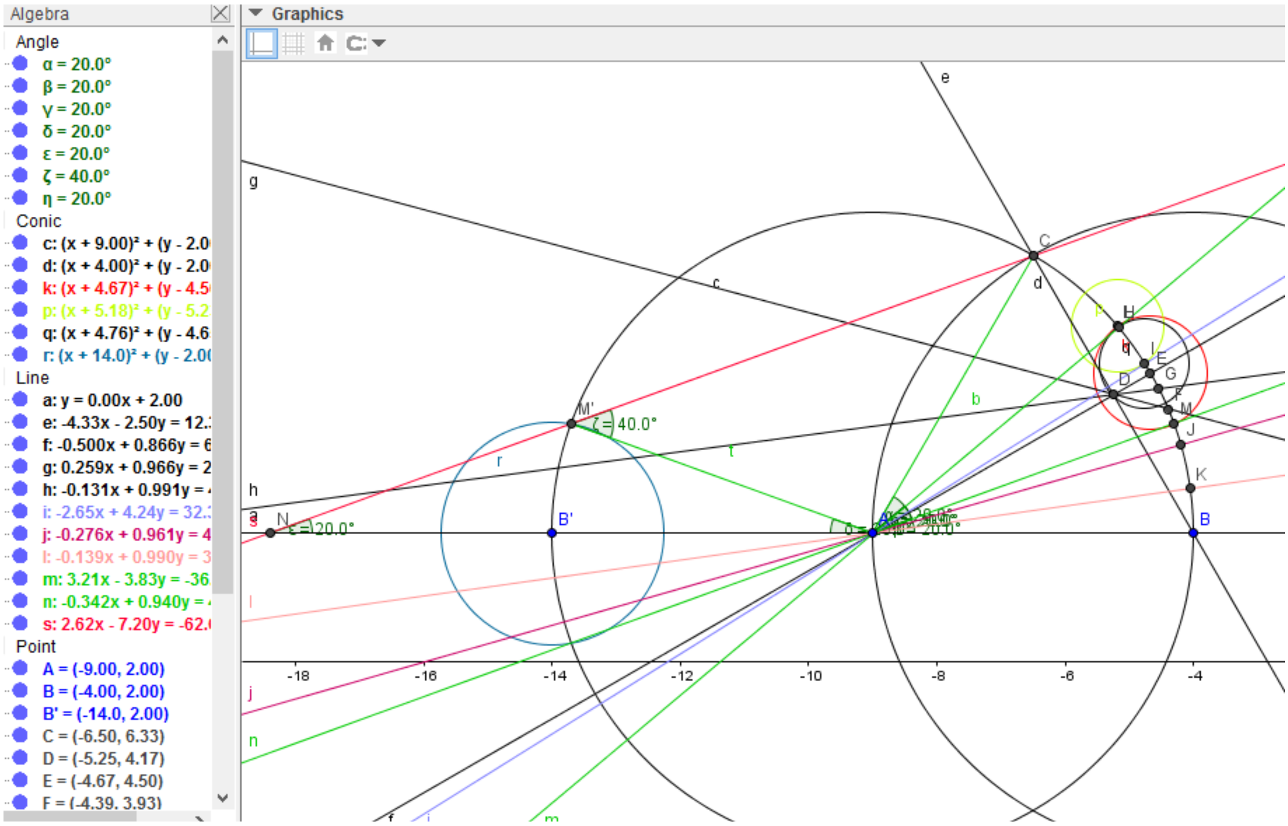

Fig. 10 Justification of the Angle Trisection Proof.

(12) Mark equal intervals of length $\overline{BM}$ along the circumference to produce figure (11).

From the construction it is observed that, the chord $\overline{BM}$ equally stroked 5 times along the circumference to produce the regular Pentagon. Let the subtended angle MAB be θ and the $\angle ABM$ be α. Since $\overline{AB}$ $=\overline{AM}$ (radii of circle), triangle MAB is an isosceles and therefore $\angle AMB = \angle ABM = \alpha$ (base angles of an isosceles triangle). Triangle $NAM \equiv MAB$ by property; SAS. It follows that $\angle AMB = \angle AMN = \alpha$. Thus $\angle BMN = 2\alpha$ (interior angle of the regular pentagon). The size of the angle $\angle BMN$ can be found by applying the expression;

$90°(2n-4)/n = 2\alpha$ (Size of one interior angle of a regular pentagon), where n is the number of sides of the regular polygon

$$\text{Since } n = 15, \text{ it follows;} \quad (8)$$

$$90°(10-4)/5 = 2\alpha \quad (9)$$

$$\text{Therefore, } 2\alpha = 108° \text{ and } \alpha = 54° \quad (10)$$

$$\angle MAB = \theta = 180° - 2\alpha = 180° - 108° = 72° \quad (11)$$

Thus $\theta = 72°$.

4.1.1 Construction of a Nonagon to justify the construction of a **10°** angle using only a ruler and compass

(1) Draw a circumference of radius $\overline{AB}$.

(2) Mark a point C on the circumference, equidistant $\overline{AB}$ from both A and B centered at A.

(3) Join the chord $\overline{CB}$, and construct its bisection through a point D to cut curve BC at E.

(4) Construct the bisection of $\angle EDB$ to cut curve BE at F.

(5) Further, construct the bisection of $\angle EDF$ to cut curve FE at G.

(6) Position the pair compass at point E and make an arc of length $\overline{DF}$ to cut curve EC at H.

(7) With the compass at point H, mark point I of distance $\overline{DG}$ between carve HE.

(8) Construct the bisection of $\angle IAB$ to cut curve BI at J.

(9) Bisect $\angle JAB$ at K on curve BJ.

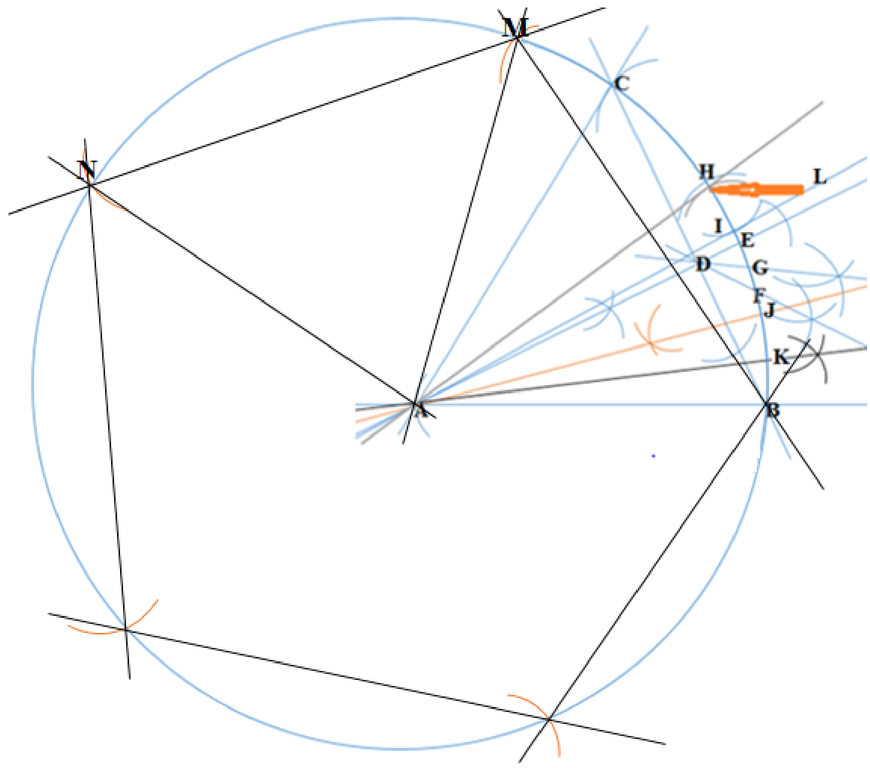

Fig. 11 Classical Construction of a Pentagon.

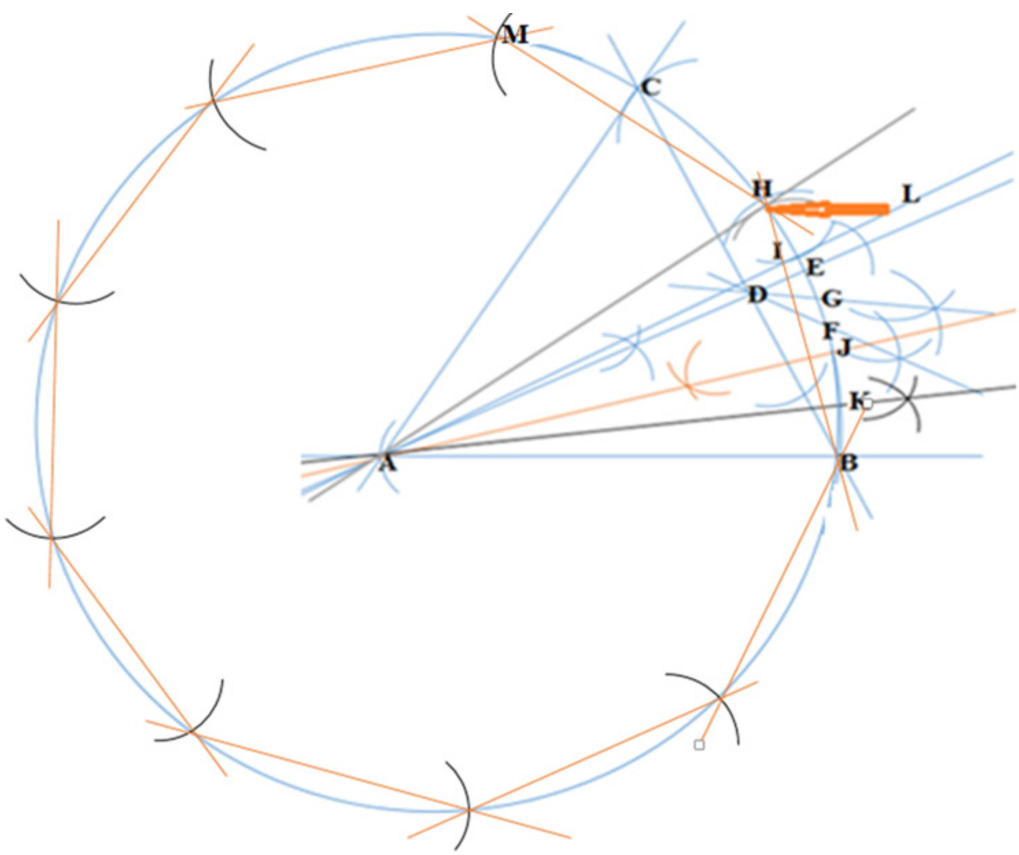

Fig. 12 Geometrical Construction of a Nonagon.

(10) Placing the compass at I, mark a point L distance $\overline{BK}$ on curve IH.

(11) Make equal intervals of length $\overline{IB}$ along the circumference to produce the figure below.

In this case, the chord LB equally marked 9 intervals along the circumference to produce the regular nonagon. Let the subtended $\angle LAB$ be θ and also let $\angle ABL$ be α. Since lines $\overline{AB}$ and $\overline{AL}$ are equal (circle radii), triangle LAB is an isosceles and therefore $\angle ALB = \angle ABL = \alpha$ (base angles of an isosceles triangle). Triangle $MAL \equiv LAB$ by property; SAS. It follows that $\angle ALB = \angle ALM = \alpha$. Thus $\angle BLM = 2\alpha$ (interior angle of the regular nonagon). The size of the angle $\angle BLM$ can be found by applying the expression for calculating the interior angles size as;

$$90°(2n-4)/n = 2\alpha$$

(Size of one interior angle of a regular nonagon), where n is the number of sides of the regular polygon. (12)

Here, $n = 9$ and therefore using n in equation (12);

$$90°(18-4)/9 = 2\alpha = 140°. \quad (13)$$

Therefore

$$\alpha = 70°. \quad (14)$$

$$\angle LAB = \theta = 180° - 2\alpha = 180° - 140° = 40°. \quad (15)$$

Thus $\theta = 40°$ as it is expected.

5.0 Revised Errors in the Classical construction of Angles in General Paper [3]

5.1.0 Use of GeoGebra software to illustrate the fault in construction of $\mathbf{10°}$ *in the given method.*

According to the article 'Classical construction of angles in general', the author presented the following steps of constructing an angle of $50°$ [3];

(1) Draw a straight line and mark two points O and P on the line.

(2) Mark a point Q equidistant $\overline{OP}$ from both O and P and draw an arc PQ centered at O.

(3) Join the chord $\overline{QP}$, and draw its bisector through a point R to cut curve PQ at S.

(4) Draw the bisector of angle SRP to cut the curve PS at T.

(5) Place your compass at point Q and make a small arc of length $\overline{RT}$ to cut curve SQ at U.

(6) Join the point U to O and there you have $\angle UOP = 50°$.

The steps were carried out for; compass and ruler construction, and also using the GeoGebra software as illustrated in figures (13) and (14).

From figure (13), the author did not provide a proof that the method would provide an angle of exactly10°. In figure (14), it is clear that $\angle UOP = 49.78°$ and not $\angle UOP = 50°$ as stated in the article. This implies that the generated chord $\overline{RT}$ subtends an angle of $10.22°$ at point O, and not $10°$ as discussed.

5.1.1 ***Use of GeoGebra software to illustrate the mistake in construction of*** $\mathbf{8°}$ ***from the given method.***

In his work, the author described how to produce an angle of $\mathbf{8°}$ using the traditional drafting tools of Greek's geometry. The following is the procedure of construction presented in the work;

(1) Draw a straight line between two points O and P.

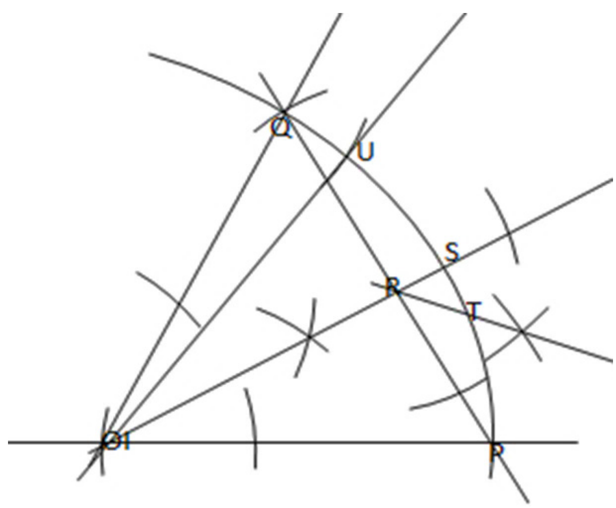

Fig. 13 Wrong illustration of constructing $50°$ angle for compass-ruler construction.

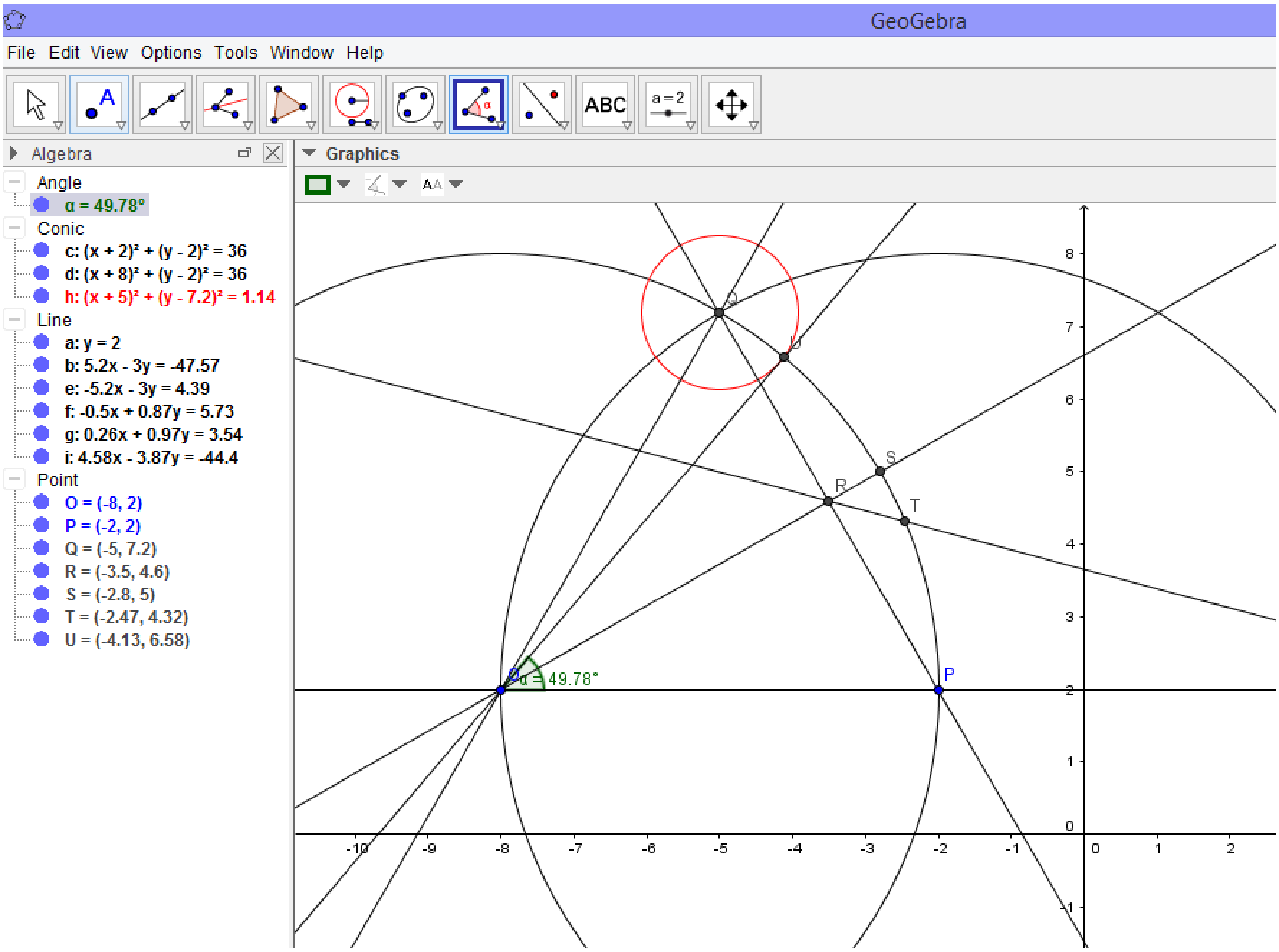

Fig. 14 **Illustration that the size of $\angle UOP = 49.78°$ and not $50°$ using GeoGebra Software**

(2) Mark a point Q equidistant $\overline{OP}$ from both O and P and draw an arc PQ centered at O.

(3) Join the chord $\overline{QP}$, and construct its bisector through a point R to cut curve PQ at S.

(4) Construct the bisector of angle SRP to cut curve PS at T.

(5) Further, construct the bisector of angle SRT to cut curve ST at U.

(6) Position the pair compass at point P and make an arc of length $\overline{RU}$ to cut curve $PTUS$ at V. $\angle VOP = 8°$

The construction was performed for ruler-compass construction and also using the GeoGebra software and the results presented in figures (15) and (16) respectively.

From figure (16), the obtained results does not correctly agree with the statement that the generated chord $\overline{RU}$ would always subtend an angle of $8°$ at a point. Contrary to figure (16), it is evident from the construction that the equivalent angle of chord $\overline{RU}$ is $8.22°$ and not exactly $8°$ as presented.

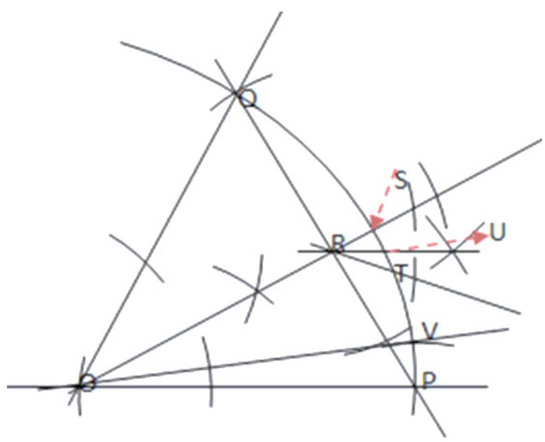

Fig. 15 **The incorrect presentation of constructing $8°$ angle for compass-ruler construction.**

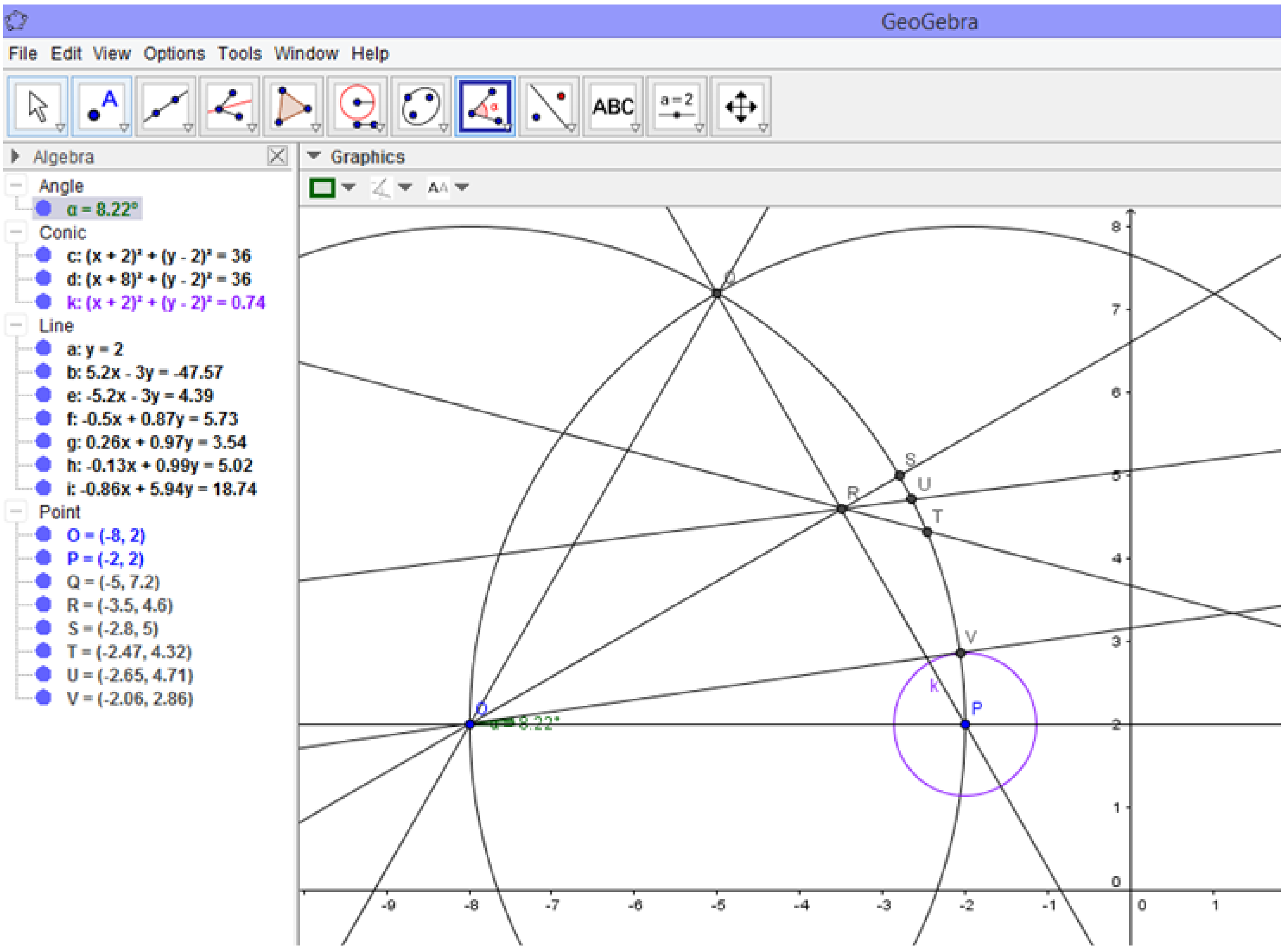

Fig. 16 Illustration that the size of $\angle VOP = 8.22°$ and not $8°$ using GeoGebra software

6. Results and Discussion

This paper has presented a flawless methodology of solving the ancient problem of angle trisection. Through the ages, mathematicians sought the trisection of an arbitrary angle but no geometrical proof has been found by this day. An ambiguous proof of the angle trisection impossibility closed the doors in solving this crucial problem by assuming it impossible [1, 2, 4, 12-15]. However, mathematicians and other practitioners still crack their minds to have the problem solved under the stated restrictions of Greek's geometry. It has well been defined that there exists a considerable ration between any two angles at a difference of $10°$ from each other, but this novel reflection was drawn in an erratic manner with no justified proof [3]. This present proof concerns revealing a correct geometric theorem of trisecting an arbitrary angle, and its precision confirmed by the trisection of $48°$ and $60°$ angles. A proposition governed by use of compass and ruler is presented contrary to the Archimedes theorem of having a marked straightedge notched in two places [24]. Through this work, it has been shown that, the general consideration of angle trisection solution as a cubic equation solution, genetically corresponds to solving the trisection of an angle in solid geometry. Geometrically, an angle is defined by two rays with a common endpoint, and only a solid angle can be solved in a $3D$ consideration. Some algebraic irrationalities are constructible in plane geometry as stated before, and the fact that a straight line segment can be proportionally fragmented and its proof did not concern degree three polynomials, shows the uncertainty in the Pierre Wantzel's proof of impossibility. Thus the cubic equation $x^3 - 3x - 1 = 0$ is not geometrically precise. GeoGebra 5.0 software was used to exemplify the correctness of the proposed method, and also to show the consistency of the construction for both Euclidean constructions and the computer aided design (CAD) approaches. The choice of using the open source GeoGebra as one of the interactive geometry software was because of its good geometry environment (Toolbox) compared to some other software, and its ease in application.

7. Conclusion

The problem of trisecting an angle has pondered in the minds of mathematicians since the antiquity, but no geometric algorithm has been made to solve the problem. Most of the presented methodologies bend the stated rules, and none has met the desired level of accuracy [22]. This novel work presents a method of trisecting an angle using the traditional Greek's tools of geometry and its precision depicted in the construction of some regular polygons which could not been correctly constructed under the set limits [30]. This paper presents a reasonable proof of redressing the problem, against the early consideration of impossibility in slicing an angle into desired fraction. From the achieved results it can reasonably be concluded that, it is geometrically possible to fraction an angle to the required proportion. In the work, an attempt has been made to bring out the misconception of the ancient problem, by defining the general algebraic error in considering the trisection of an angle as a cubic problem, from the presented impossibility proof. The construction of 2° angle implies the possibility to construct all the angles measurable using the protractor and their multiples or sub multiples, as discussed [3]. Thus the problem of trisecting an arbitrary angle, or the partitioning of a given angle into a certain ratio and vice versa is possible for compass-ruler construction. The revealed approach is contained in the formal Greek's rules of geometry.

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