WEAKLY SIGN SYMMETRIC P_0-MATRIX COMPLETION PROBLEM FOR PATTERNS OF DIGRAPHS OF ORDER 5 WITH UP TO 5 ARCS

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DECLARATION

This Thesis is my original work and has not been presented for a degree in any other institution.

Signed …… Date...……...……...

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DECLARATION BY SUPERVISORS

This Thesis has been presented with our approval as university supervisors.

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ABBREVIATIONS, SYMBOLS AND ACRONYMS.

a_{ij}: Denotes a specified entry in the partial matrix corresponding to the arc ij within the digraph

d_{ii}: Signifies the diagonal entry in the partial matrix extracted from the digraph.

n: The order of the digraph, indicating the number of vertices.

Wss P_0-matrix: Weakly sign symmetric P_0-matrix.

x_{ij}: Represents a non-specified entry in the partial matrix extracted from the digraph, indicating a missing arc between vertices i and j.

OPERATIONAL DEFINITION OF TERMS

Digraph: is a directed graph where edges are directed (Mutembei *et al*.2015).

Order of a Digraph: The number of vertices in the digraph (Choi *et al*. 2003).

Acyclic digraph: is a directed graph without any directed cycles (Harary *et al*. 2018).

Cyclic digraph: is a graph that contains at least one directed cycle which starts and end at the same vertex (Bowers *et al*.2006).

Matrix completion: is the process of filling in the missing entries of a matrix using known entries while maintaining certain structural properties (Johnson *et al*.2000).

Partial matrix: is a matrix with some entries missing or unknown (Choi *et al*.2003).

Principal minor: is the determinant of a square submatrix obtained by deleting the same set of rows and columns from larger matrix (Hogben *et al*. 2001).

Zero completion: is filling in missing entries of a partial matrix with zeros (Jumaa *et al*.2003).

Wss P_0-matrix: a matrix is Wss P_0-matrix if the off-diagonal elements have the property that if the entry in row i and column j is non-zero then the entry in row j and column i must have same sign or be zero, holding true for all pairs of indices i and j (Tomno *et al*.2028).

ABSTRACT

Matrix completion problem involves determining whether or not a completion of a partial matrix exist for a certain class of matrices satisfying a number of prescribed properties or not. Research on completion of various classes of matrices including P-matrices, P_0-matrices, as well as Wss P_0-matrices has been done. In particular, completion of Wss P_0-matrix for 4×4 matrices have been explored using digraphs with 4 arcs. However, the case of digraphs of order 5 with up to 5 arcs has not been studied. In this study, therefore, completion for non-isomorphic digraphs of order 5 with up to 5 arcs were determined. Digraphs were utilized to create partial Wss P_0-matrix from which all principal minors were obtained. Partial matrices were extracted from non-isomorphic digraphs. Principal sub-matrices were extracted from each partial matrix thereby finding the determinant of each sub-matrix obtained. Zero completion was done to all partial matrices to ascertain the viability of completion for each partial matrix. Digraphs characteristics, which leads to completion or non-completion, were analyzed. These digraphs characteristics were derived from digraphs used to construct the partial Wss P_0-matrix. This study established that all Cyclic and acyclic digraphs of order 5 with up to 5 arcs were found to have zero completion into Wss P_0-matrix. Digraphs of order 5 with 2 arcs which have positionally symmetric cycle were found to have completion. However, those digraphs of order 5 with 4 and 5 arcs which possess positionally symmetric cycle were discovered to have no completion. Insights gained from this class of matrix could be applied to fill gaps in data surveys, and business analytics, allocating resources, network modelling, optimizing processes.

CHAPTER ONE: INTRODUCTION

1.1 Background of the Study

Matrices are key tools in mathematics, used to organize numbers in rows and columns, and they play a major role in linear algebra. They allow for various operations, like addition, multiplication, and matrix decomposition. One important concept when working with matrices is matrix completion, which is a technique aimed at inferring missing values within a matrix using known elements in the matrix. This is helpful in fields like data analysis and optimization, where there is need to fill in incomplete data. Over time, matrix completion has become an important research area with problems like P-matrix completion, P_0-matrix completion and weakly sign symmetric P_0-matrix. The matrix completion problem regarding the classes of P-, P_0- were first studied by (Johnson *et al*., 2000). Completion problem regarding the classes of P-, *P0*-matrix were first studied by (Johnson, 2000). This has led interest in completion problems for weakly sign symmetric *P0*-matrix completion weakly sign symmetric P_0-matrix. All symmetric patterns specifying acyclic digraphs of order 4 without an arc (null graph), 2 arcs and regular acyclic digraphs with 4 arcs have zero completion into weakly sign symmetric P_0-matrix (Tomno,2018). However, the case of digraphs of order 5 with up to 5 arcs has not been studied. Therefore, this study focused in filling the gap determining completion of Wss P_0-matrix of order 5.

A square matrix A is one with both its rows and columns equal in number. A square matrix is considered a P_0-matrix when all its determinants are non-negative. A partial P_0-matrix refers to a partial matrix where all completely defined principal submatrices are non-negative. A sub-matrix is derived from a square matrix by removing certain rows and columns. A sub matrix that has no unspecified entry is said to be fully specified (Tomno,

2018). Let α be a subset of A = {1,2 ...., n} the principal submatrix was obtained by deleting all columns and rows not in α from A is denoted as A (α). The determinant of this principal submatrix is known as principal minor. Similarly, P(α) represents the principal sub pattern obtained from the pattern P by removing all positions where the first and second coordinates are not in α. A P_0-matrix A is classified as a weakly sign symmetric P_0-matrix if $a_{ij}\, a_{ji} \geq 0$ for all i and j (Sinha *et al*., 2019). Furthermore, the concept of weakly sign symmetric P_0-matrix adds an additional layer of complexity to matrix structures. A weak sign symmetric matrix is therefore a matrix that allows the off-diagonal elements to have the property that if the element at position (i, j) is not zero ($i \neq j$), then the element at position (j, i) must either have the same sign or be zero. Therefore, a non-negative weakly sign symmetric P_0-matrix is matrix where all entries are non-negative and exhibits weak sign symmetry. A partial matrix is considered partially weakly sign symmetric P_0-matrix if the determinant of all completely defined principal sub-matrices are non-negative and $a_{ij}\, a_{ji} \geq 0$ for all specified entries (DeAlba *et al*., 2003).

Graphs and digraphs have been used in matrix completion of various classes of matrices. A graph, denoted as $G = (V_G, E_G)$ consists of a non-empty, countable group of positive integers as vertices V_G, along with a set of unordered pairs {u, v} of vertices called edges. Given a graph $G= (V_G, E_G)$ then a graph $H= (V_H, E_H)$ qualifies as a subgraph of graph of G if V_H is a subset of V_G and E_H is a subset of E_G. A null graph is a graph with no edges (Harary, 2018). A digraph, represented as $D= (V_D, E_D)$, is essentially a graph G but with ordered pairs (u, v) of vertices indicating directed edges/arcs, where u is the initial vertex and v is the terminal vertex. A sub-digraph $H= (V_H, E_H)$ of digraph D is defined if V_H is a subset of V_D and E_H is a subset of E_D (Jumaa *et al*., 2003). The order of digraph D, denoted as |D|, refers to the

count of its vertices. A digraph is deemed complete when it contains all feasible arcs among its vertices (Choi *et al*., 2003). Multiple arcs refer to two or more arcs, while an arc connecting a vertex to itself is called a loop. A directed graph with at least one directed cycle is termed as cyclic digraph, whereas one without any cycle is called an acyclic digraph. The order of a digraph is the number of vertices it contains, while its size refers to number of arcs. A digraph in which every pair of distinct vertices is connected by a direct edge in at least one direction, is called a clique (Bowers *et al*., 2006). The degree of vertex refers to the count of edges connected to it. A graph is considered regular if all its vertices have same degree (Harary *et al*., 2018). A digraph D and D` are isomorphic if there exists a bijective homomorphism φ mapping a vertex from D to D`, such that the presence of an edge (u, v) in D corresponds to an edge (φ(u), φ(v)) in D`. This means that both digraphs have the same underlying graph structure and direction of arcs (Choi *et al*., 2003). In the context of digraphs, the presence of directed edges, or arcs, signified a directional relationship between vertices, allowing for a more nuanced representation of relationships.

Within this framework, the notion of subgraphs and sub-digraphs arises, wherein subsets of vertices and edges were extracted from the original graph or digraph. Null graphs, devoid of edges, and complete digraphs, containing all feasible arcs between vertices, represent extremes within the spectrum of digraph structures. Moreover, the presence or absence of cycles within a digraph distinguished between cyclic and acyclic digraphs, each with its unique properties and implications.

In matrix completion, patterns played a crucial role in identifying positions within matrices where completion is feasible. A pattern Q for n × n submatrices involve identifying positions within an n × n matrix subset, encompassing all diagonal positions. A symmetric pattern is

one where (i, j) is included if and only if (j, i) is also included. A positionally symmetric pattern for n × n matrices includes all diagonal positions that can be represented by a graph G= (V, E). Such a pattern exhibits the property that (i, j) is in the pattern if and only if (j, i) is also in the pattern (DeAlba & Hogben, 2000). Similarly, asymmetric patterns for n × n matrices, including all diagonal entries can be described using digraphs. In this case, if (i, j) is in the pattern (j, i) is not necessarily included. The class of weakly sign symmetric P_0-matrix is symmetric; hence they use the digraphs in its representation when solving weak sign symmetric P_0-matrix completion problems (DeAlba *et al*., 2003). A partial matrix defines a pattern if its specified entries correspond precisely to those positions outlined in the pattern (Choi, 2006). A pattern Q is considered a permutation similar to a pattern R if there exists a permutation φ of {1,2...., n} such that R is formed by mapping the pairs (i, j) in Q to φ(i), φ (j). When relabeling the vertices of a directed graph, thereby achieving a graph isomorphism, it aligned with executing a permutation similarity on the pattern (Hogben, 2001). Notably, the class of weakly sign symmetric P_0-matrix lends itself well to the representation using digraphs, owing to its inherent symmetry properties. After exploiting the symmetry inherent in these matrices, leveraged digraph-based representations to tackle matrix completion problems efficiently was done. Moreover, the concept of permutation similarity highlighted the role of relabeling vertices in achieving graph isomorphism, thereby facilitating the exploration of structural similarities between patterns.

This research therefore used digraphs to represent matrix sparsity patterns, applied zero completion on the partial matrices extracted from the non-isomorphic digraphs and determined whether they have completion or non-completion into Wss P_0-matrix.

1.2 Statement of the Problem

Matrix completion has been studied on weakly sign symmetric P_0-matrix specifying digraphs of 4 with up to arcs. However, the case of digraphs of order 5 with up to 5 arcs has not been studied. In this study, therefore, completion for non-isomorphic digraphs of order 5 with up to 5 arcs were determined offering insights into completion patterns.

1.3 Objectives of the Study

1.3.1 The general objective of this study is to determine weakly sign symmetric P_0-matrix completion for digraphs of order 5 with up to 5 arcs using digraphs.

1.3.2 The specific objectives of this study are:

i. To perform and evaluate Wss P_0- matrix completion on partial matrices obtained from non-isomorphic digraphs of order 5 with up to 5 arcs.

ii. To determine digraph characteristics which leads to completion.

iii. To determine the digraphs characteristics which leads to non-completion.

1.4 Significance of the Study

Insights gained from Wss P_0-matrix are applied in situations where missing values in structured data must be estimated. In healthcare, it can be used to predict a patient's missing medical test results or treatment responses based on available information from other patients with similar profiles. In mathematics, matrix completion is also important in solving eigen value and singular value decomposition (SVD) problems. These class of matrix, therefore, could be used to represent systems where a certain relationship between variables has a consistent sign pattern, stability analysis and network modelling.

CHAPTER TWO: LITERATURE REVIEW

Matrix completion has been the focus of extensive research for many years, driven by its practical applications in various domains such as seismic data reconstruction, image reconstruction, market surveys, and weather forecasting. While the primary goal of matrix completion is to estimate missing data points within a matrix, its implications extend far beyond mere data reconstruction. Notable areas of study within matrix completion include P-matrix completion, P_0-matrix completion, and weak sign symmetric P_0-matrix completion problems, each presenting unique challenges and avenues for exploration.

Early studies in matrix completion laid the groundwork for subsequent research endeavors. Johnson *et al*. (1994) investigated P- and P0-matrices and discovered that any graph and any order 3 digraph has P-completion. Fallat *et al*. (2000) did research on P-matrix completion under weak symmetry assumptions. In their findings they discovered that a combinatorial symmetric partial positive P-matrix has a positive P-matrix completion if the graph of its specified entries is an n-cycle. The also showed that a combinatorial symmetric partial π- matrix has a π-matrix if the graph of its specified entries is a 1-chordal graph and that this condition is also necessary for weakly sign symmetric P_0-matrix. Johnson *et al*. (2000) provided a comprehensive survey of matrix completion problems, focusing on positive definite completion, rank completion, and contraction completion.

In recent years, graphs and directed graphs have proven effective in solving matrix completion problems. Researchers have analyzed patterns through the utilization of these graphs and digraphs.

Choi *et al*. (2003) studied that a pattern of 4×4 matrices that includes all diagonal position has non-negative P_0-completion if and if its digraph is complete when it has 4-cycle. It was also discovered that any positionally symmetric pattern that includes all diagonal positions and whose graph is an n-cycle has non-negative P_0-completion if and only if n≠4.

Dealba *et al*. (2003) studied the weakly sign symmetric P-matrix completion problem and discovered that a partial sign symmetric P-matrix, whose digraph of specified entries is a symmetric n-cycle with $n \geq 6$, can be completed to a sign symmetric P-matrix establishing also the analogous completion property for a partial weakly sign symmetric P-matrix.

Hogben *et al*. (2003) extended the work of (Johnson, 1990) by proving that a larger class of patterns has P-completion including any 4×4 pattern with 8 or fewer off-diagonal positions hence proving that indeed all patterns having 8 or fewer arcs are classified as having P- completion. The concept of P-matrix was extended to P_0-matrix completion and a study carried out on digraphs of order 3 with 3 arcs. Results indicated that all 3×3 matrices have non-negative P_0-completion.

Jumaa *et al*. (2003) did research on non-negative P_0-matrix completion problem for 5×5 matrices specifying order 5 with 3 arcs and established that all non-negative P_0-matrices of 5×5 matrices specifying digraphs of order 5 with 3 arcs have zero completion. These digraphs are cyclic, acyclic, and those that have a sub-digraph which is clique.

Bowers *et al*. (2005) did research on completion problems for various classes of P-matrices. In this research all patterns of positions of n ×n matrices $n \leq 4$ were classified to whether or not every partial π-matrix can be completed to a π-matrix for any class of

positive P-, non- negative P-, or Fischer matrices. It was also found out that any order 6 graph that is block clique has a Fischer completion and any order 6 graph that contains double triangle, (symmetric) 4-cycle, or (symmetric) 5-cycle as an induced subgraph does not have Fischer completion.

Harary *et al*. (2006) discovered that it is possible to work out the number of digraphs (with n-points and q lines), therefore developing a technique for working out the no. of digraphs (with n points and q lines). The study discovered that there are 61 non-isomorphic digraphs with 5 points and 4 arcs. Out of this, 5 are cyclic while 56 are acyclic. All the digraphs are assumed to include all diagonals positions.

Munyiri *et al*., (2014) introduced a fundamentally new approach and discovered that all digraphs for P=5, Q=3 specifying 5×5 partial matrices which are either cycles or acyclic have non-negative P_0-completion.

Kumar and Ponmudi (2015) conducted research on 6×6 matrices. The study established that 6 × 6 matrices, which specify digraphs of order 6 with $q = 4$ that are acyclic or cyclic have nonnegative P_0-completion. It is achievable to find the exact number of non-isomorphic digraphs having q arcs and p vertices by a technique developed by Harary (2018). The results obtained from the technique are indicated in the table below p = 5 vertices and q arcs;

Table 2. 1

A 6×6 matrices with digraphs of order 6 with q = 4

***q*=No. of arcs**	0	1	2	3	4	5	6	7	8	9	10	$q \geq 11$
No. of digraphs	1	1	5	16	61	154	379	707	1155	1490	1670	3969

Source: *Kumar & Ponmudi (2015)*

There are 154 non-isomorphic digraphs with 5 arcs. This study focused on digraphs of order 5 with up to 5 arcs and determined which of them form partial matrices that have Wss P_0-- matrix completion or not.

Further research on matrices of at most order 4 was done by Sarma (2015). The research established that the property of non-negative P_0--completion is not inherited by induced sub- digraphs. It is therefore possible for a digraph to have non-negative P_0--completion and its sub-digraphs not to have completion.

Mutembei *et al*. (2015) did research on positive $P_{0,1}$ matrix completion problem for digraphs of order 3 with zero, one, two and three arcs and discovered that a null graph of order three has positive $P_{0,1}$ completion which has also shown that digraphs of order 3 with 1 or 2 arcs have positive $P_{0,1}$ completion. Consequently, the findings have also shown that 3 digraphs with 3 arcs which are either cyclic or acyclic have $P_{0,1}$ completion. Harary *et al*. (2018) carried out on all the other remaining 3-cycle digraphs and found that they have non-negative P_0-completion.

Tomno (2018) did research on weakly sign symmetric *Wss* P_0-matrix completion problem for symmetric patterns of acyclic digraphs of order 4 where they discovered and concluded that all symmetric patterns specifying acyclic digraphs of order 4 without an

arc (null graph), 2 arcs and regular acyclic digraphs with 4 arcs have zero completion into weakly sign symmetric P_0-matrix.

Entner *et al*. (2020) studied completion of a partial matrix to positive semi-definite matrix. The study established that a graph G is completable only if it is chordal graph. The study also found that a partial matrix A can only be completable to a positive semi-definite matrix if every specified principal sub-matrix of A is positive semi-definite.

Paul *et al*. (2021) did research on non-negative P_0-matrix completion problem for 5×5 matrices specifying cyclic digraphs with 5 vertices and 4 arcs and established that all acyclic digraphs of a 5×5 matrix with Q=4 have non-negative P_0-completion.

Tsitsiashvili *et al*. (2021) conducted research on new applied problems in the theory of acyclic digraphs and came up with two optimization problem on acyclic digraph analysis. The first problem consists of determining the minimum in terms of volume set of arcs. The second problem was to determine the smallest set of arcs, the introduction of which into an acyclic digraph turn into a strongly connected one. The first problem was solved by reduction to the problem of the maximum flow and the minimum section, while the second challenge was solved by calculating the minimum number of input arcs and determining the smallest set of input arcs in terms of minimum arc coverage of an acyclic digraph, where the solution of this problem extends to an arbitrary digraph by isolating the components by cyclic equivalence in it and the arcs between them.

However, the case of digraphs of order 5 with up to 5 arcs has not been studied. This study therefore, focused in bridging this gap by determining weakly sign symmetric P_0-matrix completion for non-isomorphic digraphs of order 5 with up to 5 arcs.

CHAPTER THREE: RESEARCH METHODOLOGY

In the process of constructing a partial P_0-matrix, digraphs were used where vertices correspond to diagonal entries d_{ii}. An arc a_{ij} in the digraph represents a specified entry in the partial matrix, while a missing arc x_{ij} represents an unspecified entry. A special feature of the digraph is when there is a double line between two vertices, say i and j. This double line represents two directed arcs: one from vertex i to vertex j and another from vertex j to vertex i. A double line in the digraph denotes a clique or a symmetric cycle.

All principal submatrices were derived from the partial Wss P_0-matrix. This class of matrices allows 0 as an entry, and by definition, diagonal entries are non-negative or (≥ 0). Principal minors of each principal submatrix were obtained. Completion was done using zero completion method where unspecified entries were set to 0. Diagonal entries were chosen to be sufficiently large while the off-diagonal entries were chosen to be small.

Furthermore, if a principal sub-matrix is completely specified it is considered a Wss P_0-matrix by the process of matrix completion. The principal minors of a matrix are the determinants of its principal sub-matrices. To check if a partial matrix has Wss P_0-completion its principal minors were examined. If all the principal minors are non-negative or (≥ 0), then the partial matrix was confirmed to be a Wss P_0-matrix, and it has zero completion to a Wss P_0-matrix.

However, if any of the principal minors is negative or (< 0), then the partial matrix cannot be completed into a Wss P_0-matrix. After determining which partial matrices can be completed into Wss P_0-matrices and which cannot, the characteristics of corresponding digraphs used to construct these partial matrices were analyzed. The characteristics of these digraphs

provide important insights into whether a matrix will have a successful Wss P_0- completion or not.

In summary, the process involved creating matrices by starting with a partially specified matrix where certain entries are unknown. By analyzing the associated digraph and calculating the principal minors, zero completion method was applied to determine whether the matrix could be completed into a Wss P_0-matrix or not. The key criterion for this completion is that all principal minors must be non-negative. If this condition holds, the partial matrix can be completed successfully; otherwise, it cannot be completed into a Wss P_0-matrix.

CHAPTER FOUR: RESULTS AND DISCUSSION

4.1 Wss p_0 -matrix completion on partial matrices obtained from Non-Isomorphic Digraphs of Order 5 with up to 5 Arcs

According to the technique developed by Harary *et al. (*2018), there are one hundred and fifty-four (154) non-isomorphic digraphs of order 5 with 5 arcs. A number of digraphs were considered and checked if they have completion or not by determining whether the determinants of the principal sub-matrices are all non-negative or not. All the directed graphs are assumed to include all slant positions.

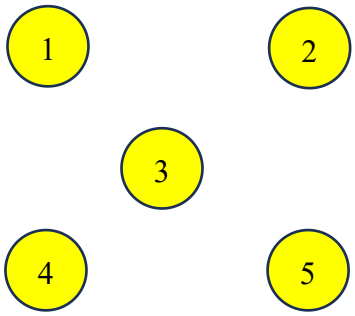
4.1. Null Graph

Example 1

Assume digraph D = {(1,1) (2,2), (3,3), (4,4), (5,5)} with 5 vertices and no arc given by;

Figure 4. 1

Null Graph



Source: Researcher (2024)

The matrix that partially outlines above digraph D is $A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

A (1, 2) $=\begin{bmatrix} d_{11} & x_{12} \\ x_{21} & d_{22} \end{bmatrix}$, A (1, 3) $= \begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$ A (1, 4) $= \begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$,

A (1, 5) $= \begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, A (2, 3) $= \begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$,A (2, 4) $=\begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$,

A (2, 5) $= \begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, A (3,4) $= \begin{bmatrix} d_{33} & x_{34} \\ x_{43} & d_{44} \end{bmatrix}$, A (3,5) $= \begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}$,

A (4,5) $= \begin{bmatrix} d_{44} & x_{45} \\ x_{54} & d_{55} \end{bmatrix}$, A(1,2,3)$= \begin{bmatrix} d_{11} & x_{12} & x_{13} \\ x_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}$, A(1,2,4) $=\begin{bmatrix} d_{11} & x_{12} & x_{14} \\ x_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}$,

A(1,2,5)$\begin{bmatrix} d_{11} & x_{12} & x_{15} \\ x_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$A(1,3,4)$= \begin{bmatrix} d_{11} & x_{13} & x_{14} \\ x_{31} & d_{33} & x_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix}$,A(1,3,5)$=\begin{bmatrix} d_{11} & x_{13} & x_{15} \\ x_{31} & d_{33} & x_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}$,

A(1,4,5)$=\begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & x_{54} & d_{55} \end{bmatrix}$A(2,3,4)$=\begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & x_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}$,A(2,3,5)$=\begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}$,

A(2,4,5) $= \begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$A (3,4,5) $= \begin{pmatrix} d_{33} & x_{34} & x_{35} \\ x_{43} & d_{44} & x_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix}$,

A (1,2,3,4) $= \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix}$,A (1,2,3,5) $= \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}$,

A(1,2,4,5) $= \begin{pmatrix} d_{11} & x_{12} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$ A(1,3,4,5) $= \begin{pmatrix} d_{11} & x_{13} & x_{14} & x_{15} \\ x_{31} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{53} & x_{54} & d_{55} \end{pmatrix}$,

A(2,3,4,5) $= \begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

Determinants of each sub-matrix above was obtained as indicated below:

Det A (1,2) = $d_{11}\, d_{22} - x_{12}\, x_{21}$

Det A (1,3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15} x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - x_{34}\, x_{43}$

Det A (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$

Det A (4 5) = $d_{44}\, d_{55} - x_{45}\, x_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}d_{33} - x_{23}\, x_{32}) - x_{12}\,(x_{21}\, d_{33} - x_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22}x_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}d_{44} - x_{24}\, x_{42}) - x_{12}\,(x_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(x_{21}\, x_{42} - d_{22}x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}d_{55} - x_{25}\, x_{52}) - x_{12}\,(x_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(x_{21}\, x_{52} - d_{22}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}d_{44} - x_{34}\, x_{43}) - x_{13}\,(x_{31}\, d_{44} - x_{34}\, x_{41}) + x_{14}\,(x_{31}\, x_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}d_{55} - x_{35}\, x_{53}) - x_{13}\,(x_{31}\, d_{55} - x_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}d_{55} - x_{45}\, x_{54}) - x_{14}\,(x_{41}\, d_{55} - x_{45}\, x_{51}) + x_{15}\,(x_{41}\, x_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}d_{44} - x_{34}\, x_{43}) - x_{23}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$

Det A (2,3,5) = $d_{22}\,(d_{33}d_{55} - x_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$

Det A (2,4,5) = $d_{22}\,(d_{44}d_{55} - x_{45}\, x_{54}) - x_{24}\,(x_{42}\, d_{55} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})$.

Det A (3,4,5) = $d_{33}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{34}\,(x_{43}\, d_{55} - x_{45}\, x_{53}) + x_{35}\,(x_{43}\, x_{54} - d_{44}\, x_{53})$

Det A (1,2,3,4) = $d_{11}[d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})]$

$- x_{12}\,[x_{21}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, x_{41})]$

$+ x_{13}\,[x_{21}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})$

$$- x_{14}\,[x_{21}\,(x_{32}\,x_{43} - d_{33}\,x_{42}) - d_{22}(x_{43}\,x_{31} - d_{33}\,x_{41}) + x_{23}\,(x_{31}\,x_{42} - x_{32}\,x_{41})].$$

$$\text{Det A } (1,2,3,5) = \quad d_{11}\,[d_{22}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - x_{23}\,(d_{55}\,x_{32} - x_{35}\,x_{52}) + x_{25}\,(x_{32}\,x_{53} - d_{33}\,x_{52})]$$

$$- x_{12}\,[x_{21}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - x_{23}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,x_{53} - d_{33}\,x_{51})]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) - d_{22}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,x_{52} - x_{32}\,x_{51})]$$

$$- x_{15}\,[x_{21}\,(x_{32}\,x_{53} - d_{33}\,x_{52}) - d_{22}\,(x_{53}\,x_{31} - d_{33}\,x_{51}) + x_{23}\,(x_{31}\,x_{52} - x_{32}\,x_{51})].$$

$$\text{Det A } (1,2,4,5) = d_{11}\,[d_{22}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{24}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{25}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$- x_{12}\,[x_{21}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{24}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{21}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - d_{22}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,x_{51})]$$

$$- x_{15}\,[x_{21}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - d_{2}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{24}\,(x_{41}\,x_{52} - x_{51}\,x_{42})].$$

$$\text{Det A } (1,3,4,5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{13}\,[x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{51}\,x_{43})].$$

$$\text{Det A } (2,3,4,5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$\text{Det (A)} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- x_{25}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))]$$

$$- x_{12}\,[x_{21}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$+ x_{23}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$- x_{24}(x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(d_{55}x_{41} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51}))$$

$$+ x_{25}(x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51}))]$$

$$+ x_{13}[x_{21}(x_{32}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(d_{55}x_{42} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52}))$$

$$- d_{22}(x_{31}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(d_{55}x_{41} - x_{45}x_{51}) + x_{35}(x_{41}x_{54} - d_{44}x_{51}))$$

$$+ x_{24}(x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(d_{55}x_{41} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51}))$$

$$- x_{25}(x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51}))]$$

$$- x_{14}[x_{21}(x_{32}(d_{55}x_{43} - x_{45}x_{53}) - d_{33}(d_{55}x_{42} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52}))$$

$$+ d_{22}(x_{31}(d_{55}x_{43} - x_{45}x_{53}) - d_{33}(d_{55}x_{41} - x_{45}x_{51}) + x_{35}(x_{41}x_{53} - x_{43}x_{51}))$$

$$- x_{23}(x_{31}(d_{55}x_{42} - x_{45}x_{52}) - x_{32}(d_{55}x_{41} - x_{45}x_{51}) + x_{35}(x_{41}x_{52} - x_{42}x_{51}))$$

$$x_{25}(x_{31}(x_{53}x_{42} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51}))]$$

$$+ x_{15}[x_{21}(x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - x_{43}x_{52}))$$

$$- d_{22}(x_{31}(x_{54}x_{43} - d_{44}x_{53}) - d_{33}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{53} - x_{43}x_{51}))$$

$$+ x_{23}(x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(x_{41}x_{54} - d_{44}x_{51}) + x_{34}(x_{41}x_{52} - x_{42}x_{51}))$$

$$- x_{24}(x_{31}(x_{42}x_{53} - x_{43}x_{52}) - x_{32}(x_{41}x_{53} - x_{43}x_{51}) + d_{33}(x_{41}x_{52} - x_{42}x_{51}))]$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}= x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 1

Determinants of 2×2 sub-matrices of "null graph"

Principal submatrix	**Principal minor**
A (1,2)	Det A (1,2) = $d_{11}\, d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}\, d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}\, d_{44} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}\, d_{55} \geq 0$.
A (2,3)	Det A (2,3) = $d_{22}\, d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}\, d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}\, d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}\, d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}\, d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 2

Determinants of 3×3 sub-matrices of "null graph"

Principal submatrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = $d_{11}\, d_{22}\, d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}\, d_{22}\, d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}\, d_{22}\, d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}\, d_{33}\, d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}\, d_{33}\, d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}\, d_{44}\, d_{55} \geq 0$.
A (2,3,4)	Det A (2,3,4) = $d_{22}\, d_{33}\, d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}\, d_{33}\, d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}\, d_{44}\, d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 3

Determinants of 4×4 sub-matrices of "null graph"

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}\, d_{22}\, d_{33}\, d_{44} \geq 0$.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}\, d_{22}\, d_{33}\, d_{55} \geq 0$.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}\, d_{22}\, d_{44}\, d_{55} \geq 0$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}\, d_{33}\, d_{44}\, d_{55} \geq 0$.
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Det A = $d_{11}\, d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Since all the determinants are non-negative then the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

4.2 Digraph of Order 5 with 1 Arc

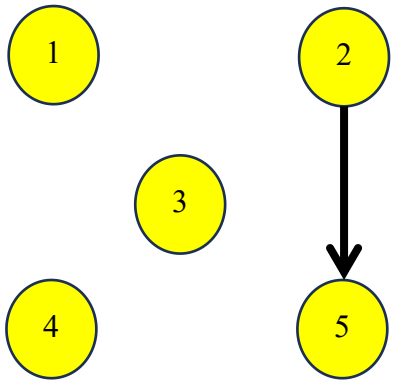
According to the technique developed by Harary *et al. (*2018), there is only one non-isomorphic digraph of order 5 with 1 arc. All other digraphs of order 5 with 1 arc are isomorphic to the digraph below.

Example 2:

Assume digraph D= {(1,1) (2,2), (2,5), (3,3), (4,4), (5,5)} with 5 vertices and 1 arc given by:

Figure 4. 2

Digraph D of order 5 with 1 arc



Source: Researcher (2024)

The partial Wss P_0-matrix specifying above digraph is A = $\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

A (1, 2) = $\begin{bmatrix} d_{11} & x_{12} \\ x_{21} & d_{22} \end{bmatrix}$, A (1, 3) = $\begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$

A (1, 4) = $\begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$, A (1, 5) = $\begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, A (2, 3) = $\begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$

A (2, 4) = $\begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$, A (2, 5) = $\begin{bmatrix} d_{22} & a_{25} \\ x_{52} & d_{55} \end{bmatrix}$, A (3,4) = $\begin{bmatrix} d_{33} & x_{34} \\ x_{43} & d_{44} \end{bmatrix}$

A (3,5) = $\begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}$, A (4, 5) = $\begin{bmatrix} d_{44} & x_{45} \\ x_{54} & d_{55} \end{bmatrix}$,

$$A(1,2,3)=\begin{bmatrix} d_{11} & x_{12} & x_{13} \\ x_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}, A(1,2,4)=\begin{bmatrix} d_{11} & x_{12} & x_{14} \\ x_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}, A(1,2,5)=\begin{bmatrix} d_{11} & x_{12} & x_{15} \\ x_{21} & d_{22} & a_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$$

$$A(1,3,4)=\begin{bmatrix} d_{11} & x_{13} & x_{14} \\ x_{31} & d_{33} & x_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix}, A(1,3,5)=\begin{bmatrix} d_{11} & x_{13} & x_{15} \\ x_{31} & d_{33} & x_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}, A(1,4,5)=\begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & x_{54} & d_{55} \end{bmatrix}$$

$$A(2,3,4)=\begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & x_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}, A(2,3,5)=\begin{bmatrix} d_{22} & x_{23} & a_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}, A(2,4,5)=\begin{pmatrix} d_{22} & x_{24} & a_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$$

$$A(3,4,5)=\begin{pmatrix} d_{33} & x_{34} & x_{35} \\ x_{43} & d_{44} & x_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix}, A(1,2,3,4)=\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix},$$

$$A(1,2,3,5)=\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{15} \\ x_{21} & d_{22} & x_{23} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}, A(1,2,4,5)=\begin{pmatrix} d_{11} & x_{12} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{24} & a_{25} \\ x_{41} & x_{42} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$$

$$A(1,3,4,5)=\begin{pmatrix} d_{11} & x_{13} & x_{14} & x_{14} \\ x_{31} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{53} & x_{54} & d_{55} \end{pmatrix}, A(2,3,4,5)=\begin{pmatrix} d_{22} & x_{23} & x_{24} & a_{25} \\ x_{32} & d_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1,2) = $d_{11}\, d_{22} - x_{12}\, x_{21}$

Det A (1,3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - a_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - x_{34}\, x_{43}$

Det A (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$

Det A (4 5) = $d_{44}\, d_{55} - x_{45}\, x_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32}) - x_{12}\,(x_{21}\, d_{33} - x_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22}\, x_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - x_{12}\,(x_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(x_{21}\, x_{42} - d_{22}\, x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}\, d_{55} - a_{25}\, x_{52}) - x_{12}\,(x_{21}\, d_{55} - a_{25}\, x_{51}) + x_{15}\,(x_{21}\, x_{52} - d_{22}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{13}\,(x_{31}\, d_{44} - x_{34}\, x_{41}) + x_{14}\,(x_{31}\, x_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{13}\,(x_{31}\, d_{55} - x_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{14}\,(x_{41}\, d_{55} - x_{45}\, x_{51}) + x_{15}\,(x_{41}\, x_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$

Det A (2,3,5) = $d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) + a_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$

Det A (2,4,5) = $d_{22}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(x_{42}\, d_{55} - x_{45}\, x_{52}) + a_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})$.

Det A (3,4,5) = $d_{33}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{34}\,(x_{43}\, d_{55} - x_{45}\, x_{53}) + x_{35}\,(x_{43}\, x_{54} - d_{44}\, x_{53})$.

Det A (1,2,3,4) = $d_{11}[d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})]$

$- x_{12}\,[x_{21}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, x_{41})]$

$+ x_{13}\,[x_{21}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$

$- x_{14}\,[x_{21}\,(x_{32}\, x_{43} - d_{33}\, x_{42}) - d_{22}(x_{43}\, x_{31} - d_{33}\, x_{41}) + x_{23}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$.

Det A (1,2,3,5) = $d_{11}\,[d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{32} - x_{35}\, x_{52}) + a_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})]$

$- x_{12}\,[x_{21}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{31} - x_{35}\, x_{51}) + a_{25}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

$+ x_{13}\,[x_{21}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) - d_{22}\,(d_{55}\, x_{31} - x_{35}\, x_{51}) + a_{25}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$

$- x_{15}\,[x_{21}\,(x_{32}\, x_{53} - d_{33}\, x_{52}) - d_{22}\,(x_{53}\, x_{31} - d_{33}\, x_{51}) + x_{23}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$.

Det A (1,2,4,5) = $d_{11}\,[d_{22}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(d_{55}\, x_{42} - x_{45}\, x_{52}) + a_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})]$

$- x_{12}\,[x_{21}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(d_{55}\, x_{41} - x_{45}\, x_{51}) + a_{25}\,(x_{41}\, x_{54} - d_{44}\, x_{51})]$

$+ x_{14}\,[x_{21}\,(d_{55}\, x_{42} - x_{45}\, x_{52}) - d_{22}\,(d_{55}\, x_{41} - x_{45}\, x_{51}) + a_{25}\,(x_{41}\, x_{52} - x_{42}\, x_{51})]$

$- x_{15}\,[x_{21}\,(x_{42}\, x_{54} - d_{44}\, x_{52}) - d_{22}\,(x_{41}\, x_{54} - d_{44}\, x_{51}) + x_{24}\,(x_{41}\, x_{52} - x_{51}\, x_{42})]$.

$$\text{Det A } (1,3,4,5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{13}\,[x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{51}\,x_{43})].$$

$$\text{Det A } (2,3,4,5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$-a_{25}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))]$$

$$- x_{12}\,[x_{21}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$+ x_{23}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$- \; x_{24}\,(x_{31}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ a_{25}\,(x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$- \; d_{22}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- \; a_{25}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$- \; x_{14}\,[x_{21}\,(x_{32}(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$+ d_{22}\,(x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$- \; x_{23}\,(x_{31}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$a_{25}\,(x_{31}\,(x_{53}\,x_{42} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$+\ x_{15}\,[x_{21}\,(x_{32}(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$-\ d_{22}\,(x_{31}\,(x_{54}\,x_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ x_{23}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$-\ x_{24}\,(x_{31}\,(x_{42}\,x_{53} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}=\ x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 4

Determinants of 2×2 sub-matrices of " digraph, order 5, 1 arc."

Principal submatrix	**Principal minor**
A (1,2)	Det A (1,2) = $d_{11}\, d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}\, d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}\, d_{44} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}\, d_{55} \geq 0$.
A (2,3)	Det A (2,3) = $d_{22}\, d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}\, d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}\, d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}\, d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}\, d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44} d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 5

Determinants of 3×3 sub-matrices of " digraph, order 5, 1 arc."

Principal submatrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = $d_{11}\, d_{22}\, d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}\, d_{22}\, d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}\, d_{22}\, d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}\, d_{33}\, d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}\, d_{33}\, d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}\, d_{44}\, d_{55} \geq 0$.
A (2,3,4)	Det A (2,3,4) = $d_{22}\, d_{33}\, d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}\, d_{33}\, d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}\, d_{44}\, d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 6

Determinants of 4×4 sub-matrices of " digraph, order 5, 1 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}\, d_{22}\, d_{33}\, d_{44} \geq 0$.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}\, d_{22}\, d_{33}\, d_{55} \geq 0$.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}\, d_{22}\, d_{44}\, d_{55} \geq 0$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}\, d_{33}\, d_{44}\, d_{55} \geq 0$.
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Det A = $d_{11}\, d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Det A (1,2) = $d_{22}\, d_{55} - a_{25}\, x_{52} \geq 0$. In addition, $a_{25}\, x_{52} = 0$ after assigning $x_{52} = 0$.

Since all the determinants are non-negative then the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

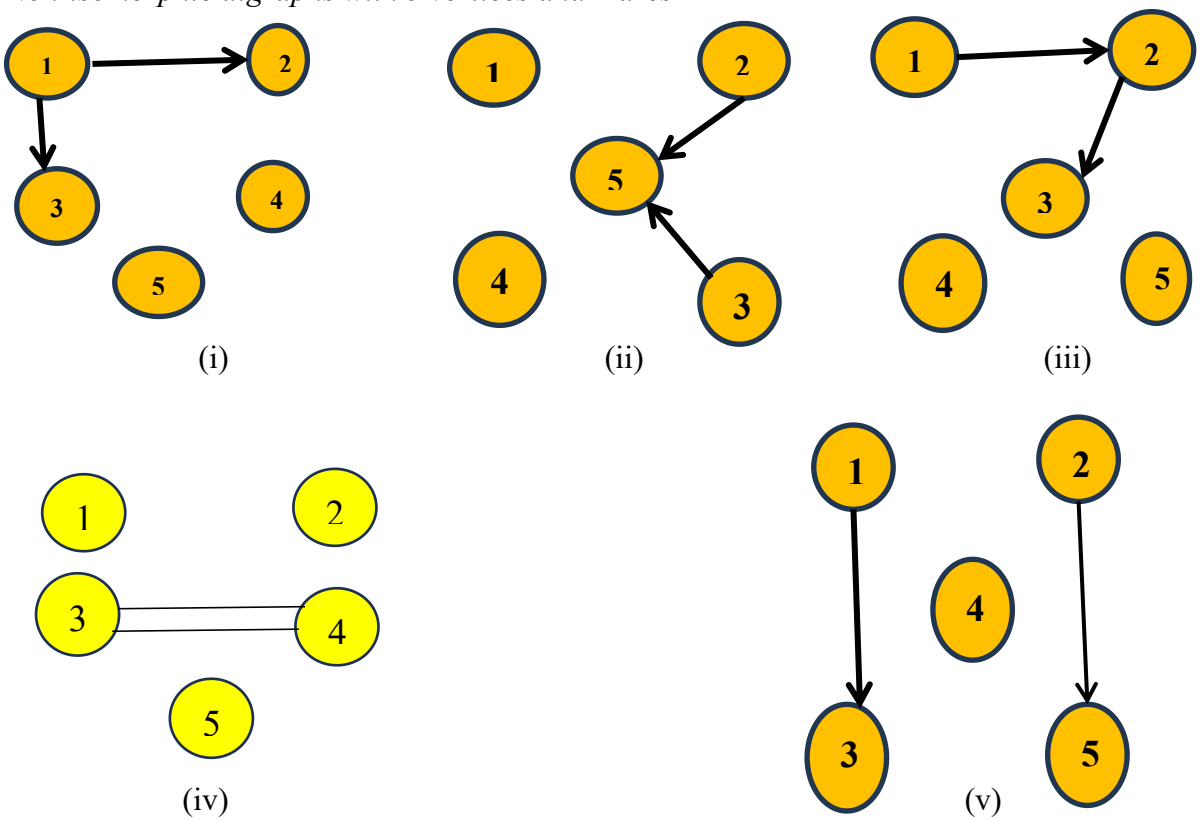
source

4.3 Digraphs of Order 5 with 2 Arcs

According to the technique developed by Harary *et al. (*2018), there are 5 non-isomorphic digraphs with 5 vertices and 2 arcs. Below are few examples of non-isomorphic digraphs with 5 vertices and 2 arcs.

Figure 4. 3

Non-isomorphic digraphs with 5 vertices and 2 arcs



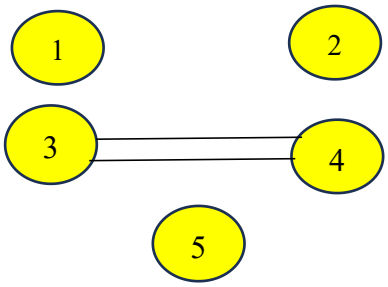
Source: Researcher (2024)

Example 3:

Assume digraph D = {(1,1), (2,2), (3,3), (3,4), (4,3) (4,4), (5,5)} with 5 vertices and 2 arcs given by:

Figure 4. 4

A digraph D of order 5 with 2 arcs



Source: Researcher (2024)

The matrix that partially outlines above digraph D is A= $\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

A (1, 2) = $\begin{bmatrix} d_{11} & x_{12} \\ x_{21} & d_{22} \end{bmatrix}$, A (1, 3) = $\begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$

A (1, 4) = $\begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$, A (1, 5) = $\begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, A (2, 3) = $\begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$

A (2, 4) = $\begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$, A (2, 5) = $\begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, A (3,4) = $\begin{bmatrix} d_{33} & a_{34} \\ a_{43} & d_{44} \end{bmatrix}$

A (3,5) = $\begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}$, A (4, 5) = $\begin{bmatrix} d_{44} & x_{45} \\ x_{54} & d_{55} \end{bmatrix}$,

A(1, 2, 3) = $\begin{bmatrix} d_{11} & x_{12} & x_{13} \\ x_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}$, A(1 ,2 ,4) = $\begin{bmatrix} d_{11} & x_{12} & x_{14} \\ x_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}$, A(1,2,5)= $\begin{bmatrix} d_{11} & x_{12} & x_{15} \\ x_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$

A(1,3,4) = $\begin{bmatrix} d_{11} & x_{13} & x_{14} \\ x_{31} & d_{33} & a_{34} \\ x_{41} & a_{43} & d_{44} \end{bmatrix}$, A (1,3,5) = $\begin{bmatrix} d_{11} & x_{13} & x_{15} \\ x_{31} & d_{33} & x_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}$, A(1,4,5) = $\begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & x_{54} & d_{55} \end{bmatrix}$

A(2,3,4) = $\begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & a_{34} \\ x_{42} & a_{43} & d_{44} \end{bmatrix}$, A(2,3,5) = $\begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}$, A(2,4,5) = $\begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$

A (3,4,5)= $\begin{pmatrix} d_{33} & a_{34} & x_{35} \\ a_{43} & d_{44} & x_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix}$, A (1,2,3,4) = $\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & a_{34} \\ x_{41} & x_{42} & a_{43} & d_{44} \end{pmatrix}$,

A (1,2,3,5) = $\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}$, A(1,2,4,5) = $\begin{pmatrix} d_{11} & x_{12} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$

$$\text{A}(1,3,4,5) = \begin{pmatrix} d_{11} & x_{13} & x_{14} & x_{14} \\ x_{31} & d_{33} & a_{34} & x_{35} \\ x_{41} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{53} & x_{54} & d_{55} \end{pmatrix}, \text{A}(2,3,4,5) = \begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & a_{34} & x_{35} \\ x_{42} & a_{43} & d_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1,2) = $d_{11}\, d_{22} - x_{12}\, x_{21}$

Det A (1,3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - a_{34}\, a_{43}$

Det A (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$

Det A (4 5) = $d_{44}\, d_{55} - x_{45}\, x_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32}) - x_{12}\,(x_{21}\, d_{33} - x_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22}\, x_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - x_{12}\,(x_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(x_{21}\, x_{42} - d_{22}\, x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52}) - x_{12}\,(x_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(x_{21}\, x_{52} - d_{22}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}\, d_{44} - a_{34}\, a_{43}) - x_{13}\,(x_{31}\, d_{44} - x_{34}\, x_{41}) + x_{14}\,(x_{31}\, x_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{13}\,(x_{31}\, d_{55} - x_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{14}\,(x_{41}\, d_{55} - x_{45}\, x_{51}) + x_{15}\,(x_{41}\, x_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}\, d_{44} - a_{34}\, a_{43}) - x_{23}\,(x_{32}\, d_{44} - a_{34}\, x_{42}) + x_{24}\,(x_{32}\, a_{43} - d_{33}\, x_{42})$

Det A (2,3,5) = $d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$

Det A (2,4,5) = $d_{22}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(x_{42}\, d_{55} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})$.

Det A (3,4,5) = $d_{33}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - a_{34}\,(a_{43}\, d_{55} - x_{45}\, x_{53}) + x_{35}\,(a_{43}\, x_{54} - d_{44}\, x_{53})$.

$$\text{Det A } (1,2,3,4) = d_{11}[d_{22}\,(d_{33}\,d_{44} - a_{34}\,a_{43}) - x_{23}\,(d_{44}\,x_{32} - a_{34}\,x_{42}) + x_{24}\,(x_{32}\,a_{43} - d_{33}\,x_{42})]$$

$$- x_{12}\,[x_{21}\,(d_{33}\,d_{44} - a_{34}\,a_{43}) - x_{23}\,(d_{44}\,x_{31} - a_{34}\,x_{41}) + x_{24}\,(x_{31}\,a_{43} - d_{33}\,x_{41})]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,d_{44} - a_{34}\,x_{42}) - d_{22}\,(d_{44}\,x_{31} - a_{34}\,x_{41}) + x_{24}\,(x_{31}\,x_{42} - x_{32}\,x_{41})]$$

$$- x_{14}\,[x_{21}\,(x_{32}\,a_{43} - d_{33}\,x_{42}) - d_{22}(a_{43}\,x_{31} - d_{33}\,x_{41}) + x_{23}\,(x_{31}\,x_{42} - x_{32}\,x_{41})].$$

$$\text{Det A } (1,2,3,5) = \quad d_{11}\,[d_{22}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - x_{23}\,(d_{55}\,x_{32} - x_{35}\,x_{52}) + x_{25}\,(x_{32}\,x_{53} - d_{33}\,x_{52})]$$

$$- x_{12}\,[x_{21}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - x_{23}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,x_{53} - d_{33}\,x_{51})]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) - d_{22}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,x_{52} - x_{32}\,x_{51})]$$

$$- x_{15}\,[x_{21}\,(x_{32}\,x_{53} - d_{33}\,x_{52}) - d_{22}\,(x_{53}\,x_{31} - d_{33}\,x_{51}) + x_{23}\,(x_{31}\,x_{52} - x_{32}\,x_{51})].$$

$$\text{Det A } (1,2,4,5) = d_{11}\,[d_{22}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{24}\,(d_5\,x_{42} - x_{45}\,x_{52}) + x_{25}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$- x_{12}\,[x_{21}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{24}\,(d_5\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{21}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - d_{22}\,(d_5\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,x_{51})]$$

$$- x_{15}\,[x_{21}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - d_2\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{24}\,(x_{41}\,x_{52} - x_{51}\,x_{42})].$$

$$\text{Det A } (1,3,4,5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_5\,a_{43} - x_{45}\,x_{53}) + x_{35}\,(a_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{13}\,[x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_5\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) - d_{33}\,(d_5\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - a_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(a_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{51}\,a_{43})].$$

$$\text{Det A } (2,3,4,5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) + x_{35}\,(a_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - a_{43}\,x_{52})]$$

$$- x_{25}\,[x_{32}\,(a_{43}\,x_{54} - d_4\,x_{53}) - d_3\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,x_{53} - a_{43}\,x_{52})].$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) + x_{35}\,(a_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(a_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - a_{43}\,x_{52}))$$

$$-x_{25}(x_{32}(a_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+a_{34}(x_{42}x_{53}-a_{43}x_{52}))]$$

$$-x_{12}[x_{21}(d_{33}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(d_{55}a_{43}-x_{45}x_{53})+x_{35}(a_{43}x_{54}-d_{44}x_{53}))$$

$$+x_{23}(x_{31}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(d_{55}x_{41}-x_{45}x_{51})+x_{35}(x_{41}x_{54}-d_{44}x_{51}))$$

$$-x_{24}(x_{31}(a_{43}d_{55}-x_{45}x_{53})-d_{33}(d_{55}x_{41}-x_{45}x_{51})+x_{35}(x_{41}x_{53}-a_{43}x_{51}))$$

$$+x_{25}(x_{31}(a_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+a_{34}(x_{41}x_{53}-a_{43}x_{51}))]$$

$$+x_{13}[x_{21}(x_{32}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(d_{55}x_{42}-x_{45}x_{52})+x_{35}(x_{42}x_{54}-d_{44}x_{52}))$$

$$-d_{22}(x_{31}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(d_{55}x_{41}-x_{45}x_{51})+x_{35}(x_{41}x_{54}-d_{44}x_{51}))$$

$$+x_{24}(x_{31}(x_{42}d_{55}-x_{45}x_{52})-x_{32}(d_{55}x_{41}-x_{45}x_{51})+x_{35}(x_{41}x_{52}-x_{42}x_{51}))$$

$$-x_{25}(x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(x_{41}x_{54}-d_{4}x_{51})+a_{34}(x_{41}x_{52}-x_{42}x_{51}))]$$

$$-x_{14}[x_{21}(x_{32}(d_{55}a_{43}-x_{45}x_{53})-d_{33}(d_{55}x_{42}-x_{45}x_{52})+x_{35}(x_{42}x_{53}-a_{43}x_{52}))$$

$$+d_{22}(x_{31}(d_{55}a_{43}-x_{45}x_{53})-d_{33}(d_{55}x_{41}-x_{45}x_{51})+x_{35}(x_{41}x_{53}-a_{43}x_{51}))$$

$$-x_{23}(x_{31}(d_{55}x_{42}-x_{45}x_{52})-x_{32}(d_{55}x_{41}-x_{45}x_{51})+x_{35}(x_{41}x_{52}-x_{42}x_{51}))$$

$$x_{25}(x_{31}(x_{53}x_{42}-a_{43}x_{52})-x_{32}(x_{41}x_{53}-a_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51}))]$$

$$+x_{15}[x_{21}(x_{32}(a_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+a_{34}(x_{42}x_{53}-a_{43}x_{52}))$$

$$-d_{22}(x_{31}(x_{54}a_{43}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+a_{34}(x_{41}x_{53}-a_{43}x_{51}))$$

$$+x_{23}(x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(x_{41}x_{54}-d_{44}x_{51})+a_{34}(x_{41}x_{52}-x_{42}x_{51}))$$

$$-x_{24}(x_{31}(x_{42}x_{53}-a_{43}x_{52})-x_{32}(x_{41}x_{53}-a_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51}))]$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}=x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54}=0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 7

Determinants of 2×2 sub-matrices of " clique digraph, order 5, 2 arc."

Principal submatrix	**Principal minor**
A (1,2)	Det A (1,2) = $d_{11}\, d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}\, d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}\, d_{44} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}\, d_{55} \geq 0$.
A (2,3)	Det A (2,3) = $d_{22}\, d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}\, d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}\, d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $(d_{33}\ d_{44}) - (a_{34}\ a_{43}) \geq 0$. (since (3,4) is completely defined)
A (3,5)	Det A (3,5) = $d_{33}\, d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 8

Determinants of 3×3 sub-matrices of " clique digraph, order 5, 2 arc."

Principal submatrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = $d_{11}\, d_{22}\, d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}\, d_{22}\, d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}\, d_{22}\, d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}(d_{33}\ d_{44} - a_{34}\ a_{43}) \geq 0$. (since (3,4) is completely defined)
A (1,3,5)	Det A (1,3,5) = $d_{11}\, d_{33}\, d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}\, d_{44}\, d_{55} \geq 0$.
A (2,3,4)	Det A (2,3,4) = $d_{22}(d_{33}\ d_{44} - a_{34}\ a_{43}) \geq 0$. (since (3,4) is completely defined)
A (2,3,5)	Det A (2,3,5) =$d_{22}\, d_{33}\, d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}\, d_{44}\, d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{55}(d_{33}\ d_{44} - a_{34}\ a_{43}) \geq 0$. (since (3,4) is completely defined)

Source: Researcher (2024)

Table 4. 9

Determinants of 4×4 sub-matrices of " clique digraph, order 5, 2 arc."

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}\ d_{22}(d_{33}\ d_{44} - a_{34}\ a_{43}) \geq 0$. (since (3,4) is completely defined)
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}\ d_{22}\ d_{33}\ d_{55} \geq 0$.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}\ d_{22}\ d_{44}\ d_{55} \geq 0$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}\ d_{55}(d_{33}\ d_{44} - a_{34}\ a_{43}) \geq 0$. (since (3,4) is completely defined)
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\ d_{55}(d_{33}\ d_{44} - a_{34}\ a_{43}) \geq 0$. (since (3,4) is completely defined) ≥ 0.

Source: Researcher (2024)

Det A = $d_{11}\ d_{22}\ d_{33}\ d_{44}\ d_{55} - a_{34}\ a_{43}\ d_{55} = d_{11}\ d_{22}\ d_{55}\ (d_{33}\ d_{44} - a_{34}\ a_{43}) \geq 0$, (since (3,4) is completely defined).

Since all the determinants are non-negative then the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

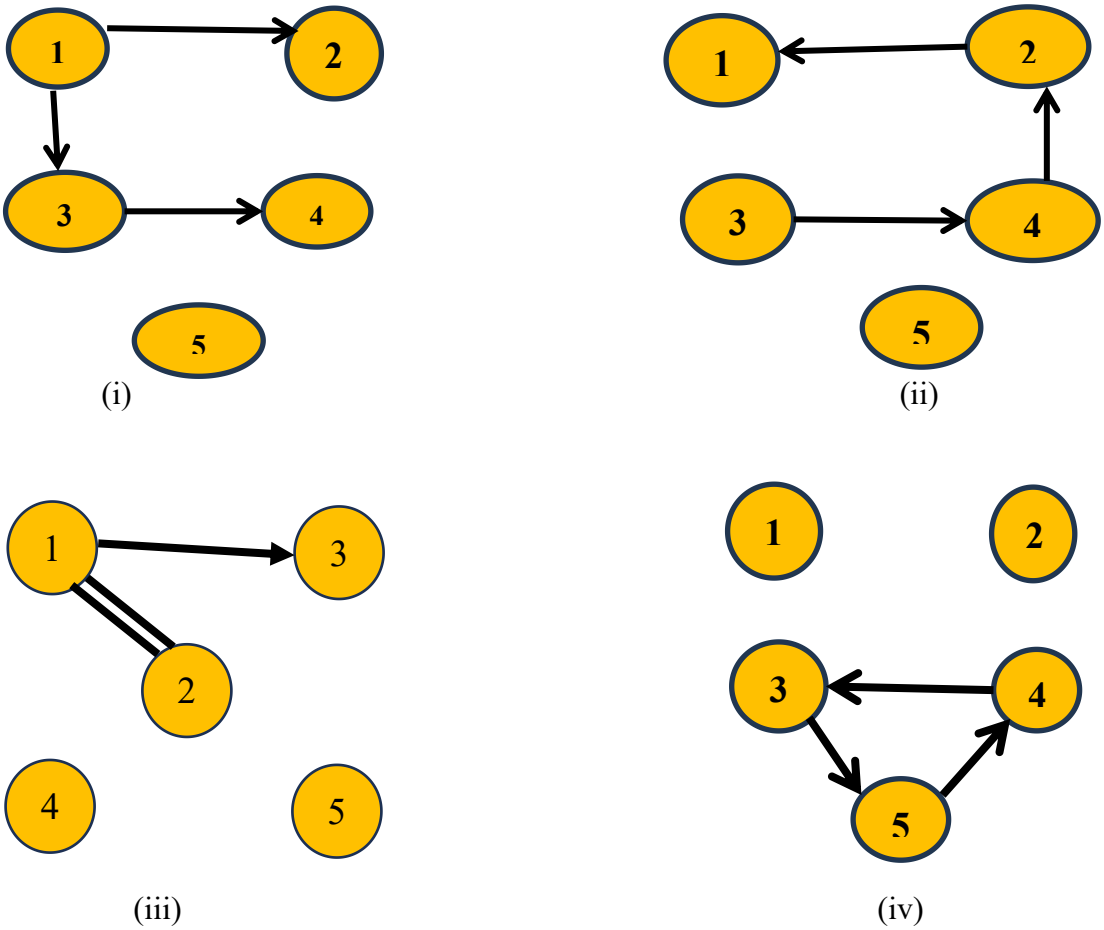
All other digraphs were solved in a similar manner and were found to have zero completion into a Wss P_0-matrix.

4.4 Digraphs of Order 5 with 3 Arcs

According to the technique developed by Harary *et al. (*2018), there are 16 non-isomorphic digraphs with 5 vertices and 3 arcs. Below are few examples of non-isomorphic digraphs with 5 vertices and 3 arcs.

Figure 4. 5

Non-isomorphic digraphs with 5 vertices and 3 arcs

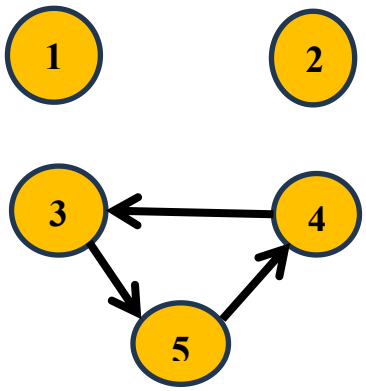


Source: Researcher (2024)

Example 4

Figure 4. 6

Cyclic digraph



Source: Researcher (2024)

The partial matrix that specifies above digraph D is $A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & a_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & a_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

$A(1, 2) = \begin{bmatrix} d_{11} & x_{12} \\ x_{21} & d_{22} \end{bmatrix}$, $A(1, 3) = \begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$

$A(1, 4) = \begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$, $A(1, 5) = \begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, $A(2, 3) = \begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$

$A(2, 4) = \begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$, $A(2, 5) = \begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, $A(3,4) = \begin{bmatrix} d_{33} & x_{34} \\ a_{43} & d_{44} \end{bmatrix}$

$A(3,5) = \begin{bmatrix} d_{33} & a_{35} \\ x_{53} & d_{55} \end{bmatrix}$, $A(4, 5) = \begin{bmatrix} d_{44} & x_{45} \\ a_{54} & d_{55} \end{bmatrix}$,

$$A(1,2,3)=\begin{bmatrix} d_{11} & x_{12} & x_{13} \\ x_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}, A(1,2,4)=\begin{bmatrix} d_{11} & x_{12} & x_{14} \\ x_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}, A(1,2,5)=\begin{bmatrix} d_{11} & x_{12} & x_{15} \\ x_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$$

$$A(1,3,4) = \begin{bmatrix} d_{11} & x_{13} & x_{14} \\ x_{31} & d_{33} & x_{34} \\ x_{41} & a_{43} & d_{44} \end{bmatrix}, A(1,3,5) = \begin{bmatrix} d_{11} & x_{13} & x_{15} \\ x_{31} & d_{33} & a_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}, A(1,4,5) = \begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & a_{54} & d_{55} \end{bmatrix}$$

$$A(2,3,4) = \begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & x_{34} \\ x_{42} & a_{43} & d_{44} \end{bmatrix}, A(2,3,5 = \begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & a_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}, A(2,4,5) = \begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & a_{54} & d_{55} \end{pmatrix}$$

$$A\ (3,4,5)=\begin{pmatrix} d_{33} & x_{34} & a_{35} \\ x_{43} & d_{44} & x_{45} \\ x_{53} & a_{54} & d_{55} \end{pmatrix}, A\ (1,2,3,4) =\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix},$$

$$A\ (1,2,3,5) = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}, A(1,2,4,5) = \begin{pmatrix} d_{11} & x_{12} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & x_{45} \\ x_{51} & x_{52} & a_{54} & d_{55} \end{pmatrix}$$

$$A(\ 1,3,4,5) = \begin{pmatrix} d_{11} & x_{13} & x_{14} & x_{14} \\ x_{31} & d_{33} & x_{34} & a_{35} \\ x_{41} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{53} & a_{54} & d_{55} \end{pmatrix}, A(2,3,4,5) = \begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & x_{34} & a_{35} \\ x_{42} & a_{43} & d_{44} & x_{45} \\ x_{52} & x_{53} & a_{54} & d_{55} \end{pmatrix}$$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1,2) = $d_{11}\, d_{22} - x_{12}\, x_{21}$

Det A (1,3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - x_{34}\, a_{43}$

Det A (3,5) = $d_{33}\, d_{55} - a_{35}\, x_{53}$

Det A (4 5) = $d_{44}\, d_{55} - x_{45}\, a_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32}) - x_{12}\,(x_{21}\, d_{33} - x_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22} x_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - x_{12}\,(x_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(x_{21}\, x_{42} - d_{22} x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52}) - x_{12}\,(x_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(x_{21}\, x_{52} - d_{2}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}\, d_{44} - x_{34}\, a_{43}) - x_{13}\,(x_{31}\, d_{44} - x_{34}\, x_{41}) + x_{14}\,(x_{31}\, a_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - x_{13}\,(x_{31}\, d_{55} - a_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}\, d_{55} - x_{45}\, a_{54}) - x_{14}\,(x_{41}\, d_{55} - x_{45}\, x_{51}) + x_{15}\,(x_{41}\, a_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}\, d_{44} - x_{34}\, a_{43}) - x_{23}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, a_{43} - d_{33}\, x_{42})$

Det A (2,3,5) = $d_{22}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - a_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$

Det A (2,4,5) = $d_{22}\,(d_{44}\, d_{55} - x_{45}\, a_{54}) - x_{24}\,(x_{42}\, d_{55} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, a_{54} - d_{44}\, x_{52})$.

Det A (3,4,5) = $d_{33}\,(d_{44}\, d_{55} - x_{45}\, a_{54}) - x_{34}\,(a_{43}\, d_{55} - x_{45}\, x_{53}) + a_{35}\,(a_{43}\, a_{54} - d_{44}\, x_{53})$.

Det A (1,2,3,4) = $d_{11}[d_{22}\,(d_{33}\, d_{44} - x_{34}\, a_{43}) - x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, a_{43} - d_{33}\, x_{42})]$

$- x_{12}\,[x_{21}\,(d_{33}\, d_{44} - x_{34}\, a_{43}) - x_{23}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, a_{43} - d_{33}\, x_{41})]$

$+ x_{13}\,[x_{21}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})$

$- x_{14}\,[x_{21}\,(x_{32}\, a_{43} - d_{33}\, x_{42}) - d_{22}(a_{43}\, x_{31} - d_{33}\, x_{41}) + x_{23}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$.

Det A (1,2,3,5) = $d_{11}\,[d_{22}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{32} - a_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})]$

$- x_{12}\,[x_{21}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{31} - a_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{53} - d_{33}\, x_{51})]$

$+ x_{13}\,[x_{21}\,(x_{32}\, d_{55} - a_{35}\, x_{52}) - d_{22}\,(d_{55}\, x_{31} - a_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$

$- x_{15}\,[x_{21}\,(x_{32}\, x_{53} - d_{33}\, x_{52}) - d_{22}\,(x_{53}\, x_{31} - d_{33}\, x_{51}) + x_{23}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$.

Det A (1,2,4,5) = $d_{11}\,[d_{22}\,(d_{44}\, d_{55} - x_{45}\, a_{54}) - x_{24}\,(d_{55}\, x_{42} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, a_{54} - d_{44}\, x_{52})]$

$- x_{12}\,[x_{21}\,(d_{44}\, d_{55} - x_{45}\, a_{54}) - x_{24}\,(d_{55}\, x_{41} - x_{45}\, x_{51}) + x_{25}\,(x_{41}\, a_{54} - d_{44}\, x_{51})]$

$+ x_{14}\,[x_{21}\,(d_{55}\, x_{42} - x_{45}\, x_{52}) - d_{22}\,(d_{55}\, x_{41} - x_{45}\, x_{51}) + x_{25}\,(x_{41}\, x_{52} - x_{42}\, x_{51})]$

$- x_{15}\,[x_{21}\,(x_{42}\, a_{54} - d_{44}\, x_{52}) - d_{2}\,(x_{41}\, a_{54} - d_{44}\, x_{51}) + x_{24}\,(x_{41}\, x_{52} - x_{51}\, x_{42})]$.

$$\text{Det A } (1,3,4,5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) + a_{35}\,(a_{43}\,a_{54} - d_{44}\,x_{53})]$$

$$- x_{13}\,[x_{31}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{53} - a_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(a_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{51}\,a_{43})].$$

$$\text{Det A } (2,3,4,5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) + a_{35}\,(a_{43}\,a_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(a_{43}\,d_{55} - x_{45}\,x_{53}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{53} - a_{43}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{53} - a_{43}\,x_{52})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52})]$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) + a_{35}\,(a_{43}\,a_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(a_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{53} - a_{43}\,x_{52}))$$

$$-x_{25}\,(x_{32}\,(a_{43}\,a_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - a_{43}\,x_{52}))]$$

$$- x_{12}\,[x_{21}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) + a_{35}\,(a_{43}\,a_{54} - d_{44}\,x_{53}))$$

$$+ x_{23}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51}))$$

$$- \; x_{24}\,(x_{31}\,(a_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{53} - a_{43}\,x_{51}))$$

$$+ x_{25}\,(x_{31}\,(a_{43}\,a_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - a_{43}\,x_{51}))]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52}))$$

$$- \; d_{22}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- \; x_{25}\,(x_{31}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$- x_{14}\,[x_{21}\,(x_{32}(d_{55}\,a_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{53} - a_{43}\,x_{52}))$$

$$+ d_{22}\,(x_{31}\,(d_{55}\,a_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{53} - a_{43}\,x_{51}))$$

$$- \; x_{23}\,(x_{31}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$x_{25}\,(x_{31}\,(x_{53}\,x_{42} - a_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - a_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$+ x_{15}\,[x_{21}\,(x_{32}(a_{43}\,a_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - a_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(a_{54}\,a_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - a_{43}\,x_{51}))$$

$$+ x_{23}\,(x_{31}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{24}\,(x_{31}\,(x_{42}\,x_{53} - a_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - a_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}= x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 10

Determinants of 2×2 sub-matrices of " cyclic digraph, order 5, 3 arc."

Principal submatrix	**Principal minor**
A (1,2)	Det A (1,2) = $d_{11}\, d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}\, d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $d_{11}\, d_{44} \geq 0$.
A (1,5)	Det A (1,5) = $d_{11}\, d_{55} \geq 0$.
A (2,3)	Det A (2,3) = $d_{22}\, d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}\, d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}\, d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}\, d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}\, d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 11

Determinants of 3×3 sub-matrices of " cyclic digraph, order 5, 3 arc."

Principal submatrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = $d_{11}\, d_{22}\, d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{11}\, d_{22}\, d_{44} \geq 0$.
A (1,2,5)	Det A (1,2,5) = $d_{11}\, d_{22}\, d_{55} \geq 0$.
A (1,3,4)	Det A (1,3,4) = $d_{11}\, d_{33}\, d_{44} \geq 0$.
A (1,3,5)	Det A (1,3,5) = $d_{11}\, d_{33}\, d_{55} \geq 0$.
A (1,4,5)	Det A (1,4,5) = $d_{11}\, d_{44}\, d_{55} \geq 0$.
A (2,3,4)	Det A (2,3,4) = $d_{22}\, d_{33}\, d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}\, d_{33}\, d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}\, d_{44}\, d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}\, d_{44}\, d_{55} + a_{35}\, a_{43}\, a_{54} \geq 0$.

Source: Researcher (2024)

Table 4. 12

Determinants of 4×4 sub-matrices of " cyclic digraph, order 5, 3 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}\, d_{22}\, d_{33}\, d_{44} \geq 0$.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}\, d_{22}\, d_{33}\, d_{55} \geq 0$.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}\, d_{22}\, d_{44}\, d_{55} \geq 0$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}\, d_{33}\, d_{44}\, d_{55} + d_{11}\, a_{35}\, a_{43}\, a_{54} \geq 0$.
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} + d_{22}\, a_{35}\, a_{43}\, a_{54} \geq 0$.

Source: Researcher (2024)

Det A = $d_{11}\, d_{22}\, d_{33}\, d_{44}\, d_{55} + d_{11}\, a_{35}\, a_{43}\, a_{54} \geq 0$.

Det A (3,5) = $d_{22}\, d_{55} - a_{35}\, x_{53} \geq 0$. Additionally, $a_{35}\, x_{53} = 0$ after assigning $x_{53} = 0$.

Correspondingly, determinants of A (3,5), A (5,4), A (4,3) are ≥ 0. Since all the determinants are non-negative then the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

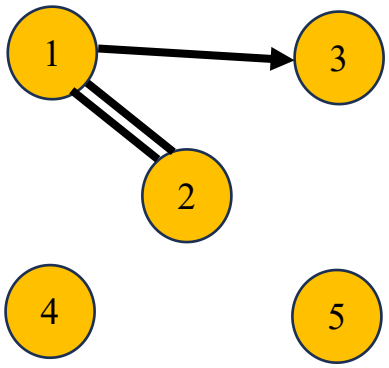
Case 2

Example 5:

Assume digraph D = {(1,1), (1,2), (1,3), (2,1), (2,2), (3,3), (4,4), (5,5)} with 5 vertices and 3 arcs given by:

Figure 4. 7

A digraph with clique sub-digraph D of Order 5 and 3 arcs



Source: Researcher (2024)

The partial matrix that specifies the above digraph A is $= \begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

$$A(1, 2) = \begin{bmatrix} d_{11} & a_{12} \\ a_{21} & d_{22} \end{bmatrix}, A(1, 3) = \begin{bmatrix} d_{11} & a_{13} \\ a_{31} & d_{33} \end{bmatrix}$$

$$A(1, 4) = \begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}, A(1, 5) = \begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}, A(2, 3) = \begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$$

$$A(2, 4) = \begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}, A(2, 5) = \begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}, A(3, 4) = \begin{bmatrix} d_{33} & x_{34} \\ x_{43} & d_{44} \end{bmatrix}$$

$$A(3, 5) = \begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}, A(4, 5) = \begin{bmatrix} d_{44} & x_{45} \\ x_{54} & d_{55} \end{bmatrix}$$

$$A(1, 2, 3) = \begin{bmatrix} d_{11} & a_{12} & a_{13} \\ a_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}, A(1, 2, 4) = \begin{bmatrix} d_{11} & a_{12} & x_{14} \\ a_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix},$$

$$\text{A (1, 2, 5)} = \begin{bmatrix} d_{11} & a_{12} & x_{15} \\ a_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix} \text{A (1, 3, 4)} = \begin{bmatrix} d_{11} & a_{12} & x_{14} \\ x_{31} & d_{33} & x_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix},$$

$$\text{A (1, 3, 5)} = \begin{bmatrix} d_{11} & a_{13} & x_{15} \\ x_{31} & d_{33} & x_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}, \text{A (1, 4, 5)} = \begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & x_{54} & d_{55} \end{bmatrix}$$

$$\text{A (2, 3, 4)} = \begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & x_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}, \text{A (2, 3, 5)} = \begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix},$$

$$\text{A (2, 4, 5)} = \begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix} \text{A (3, 4, 5)} = \begin{pmatrix} d_{33} & x_{34} & x_{35} \\ x_{43} & d_{44} & x_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix},$$

$$\text{A (1, 2, 3, 4)} = \begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} \\ a_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix}, \text{A (1, 2, 3, 5)} = \begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix},$$

$$\text{A (1, 2, 4, 5)} = \begin{pmatrix} d_{11} & a_{12} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}, \text{A (1, 3, 4, 5)} = \begin{pmatrix} d_{11} & a_{13} & x_{14} & x_{15} \\ x_{31} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{53} & x_{54} & d_{55} \end{pmatrix},$$

$$\text{A (2, 3, 4, 5)} = \begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1,2) = $d_{11}\, d_{22} - a_{12}\, a_{21}$

Det A (1,3) = $d_{11}\, d_{33} - a_{13}\, x_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - x_{34}\, x_{43}$

Det A (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$

Det A (4,5) = $d_{44}\, d_{55} - x_{45}\, x_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32}) - a_{12}\,(a_{21}\, d_{33} - x_{23}\, x_{31}) + a_{13}\,(a_{21}\, x_{32} - d_{22}\, x_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - a_{12}\,(a_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(a_{21}\, x_{42} - d_{22}\, x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52}) - a_{12}\,(a_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(a_{21}\, x_{52} - d_{22}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - a_{13}\,(x_{31}\, d_{44} - x_{34}\, x_{41}) + x_{14}\,(x_{31}\, x_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - a_{13}\,(x_{31}\, d_{55} - x_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{14}\,(x_{41}\, d_{55} - x_{45}\, x_{51}) + x_{15}\,(x_{41}\, x_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$

Det A (2,3,5) = $d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$

Det A (2,4,5) = $d_{22}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(x_{42}\, d_{55} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})$

Det A (3,4,5) = $d_{33}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{34}\,(x_{43}\, d_{55} - x_{45}\, x_{53}) + x_{35}\,(x_{43}\, x_{54} - d_{44}\, x_{53})$

Det A (1,2,3,4) = $d_{11}\,[d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})]$

$- a_{12}\,[a_{21}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, x_{41})]$

$+ a_{13}\,[a_{21}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$

$- x_{14}\,[a_{21}\,(x_{32}\, x_{43} - d_{33}\, x_{42}) - d_{22}\,(x_{43}\, x_{31} - d_{33}\, x_{41}) + x_{23}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$.

Det A (1,2,3,5) = $d_{11}\,[d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{32} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})]$

$- a_{12}\,[a_{21}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{31} - x_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{53} - d_{33}\, x_{51})]$

$+ a_{13}\,[a_{21}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) - d_{22}\,(d_{55}\, x_{31} - x_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$

$- x_{15}\,[a_{21}\,(x_{32}\, x_{53} - d_{33}\, x_{52}) - d_{22}\,(x_{53}\, x_{31} - d_{33}\, x_{51}) + x_{23}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$

Det A (1,2,4,5) = $d_{11}\,[d_{22}\,(d_{4\,4} d_{55} - x_{45}\, x_{54}) - x_{24}\,(d_{55}\, x_{42} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})]$

$- a_{12}\,[a_{21}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(d_{55}\, x_{41} - x_{45}\, x_{51}) + x_{25}\,(x_{41}\, x_{54} - d_{44}\, x_{51})]$

$$+ x_{14}\,[a_{21}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - d_{22}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,x_{51})]$$

$$- x_{15}\,[a_{21}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - d_{22}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{24}\,(x_{41}\,x_{52} - x_{51}\,x_{42})]$$

$$\text{Det A } (1,3,4,5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- a_{13}\,[x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{51}\,x_{43})]$$

$$\text{Det A } (2,3,4,5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{25}\,[x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- x_{25}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))]$$

$$- a_{12}\,[a_{21}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$+ x_{23}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$- x_{24}\,(x_{31}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ x_{25}\,(x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))]$$

$$+ a_{13}\,[a_{21}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_{4}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$- x_{14}\,[a_{21}\,(x_{32}(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$+ d_{22}\,(x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$- x_{23}\,(x_{31}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$+ x_{25}\,(x_{31}\,(x_{53}\,x_{42} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$+ x_{15}\,[a_{21}\,(x_{32}(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(x_{54}\,x_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ x_{23}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{24}\,(x_{31}\,(x_{42}\,x_{53} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))].$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}= x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 13

Determinants of 2×2 sub-matrices of " clique digraph, order 5, 3 arc."

Principal sub-matrix	**Principal minor**
A (1,2)	Det A (1,2) = $(d_{11}\,d_{22}) - (a_{12}\,a_{21}) \geq 0$ since (1,2) is completely defined.
A (1,3)	Det A (1,3) = $d_{11}\,d_{33} \geq 0$
A (1,4)	Det A (1,4) = $d_{11}\,d_{44} \geq 0$
A (1,5)	Det A (1,5) = $d_{11}\,d_{55} \geq 0$
A (2,3)	Det A (2,3) = $d_{22}\,d_{33} \geq 0$
A (2,4)	Det A (2,4) = $d_{22}\,d_{44} \geq 0$
A (2,5)	Det A (2,5) = $d_{22}\,d_{55} \geq 0$
A (3,4)	Det A (3,4) = $d_{33}\,d_{44} \geq 0$
A (3,5)	Det A (3,5) = $d_{33}\,d_{55} \geq 0$
A (4,5)	Det A (4,5) = $d_{44}\,d_{55} \geq 0$

Source: Researcher (2024)

Table 4. 14

Determinants of 3×3 sub-matrices of " clique digraph, order 5, 3 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = $d_{33}(d_{11}\,d_{22} - a_{12}\,a_{21}) \geq 0$ since (1,2) is completely defined.
A (1,2,4)	Det A (1,2,4) = $d_{44}(d_{11}\,d_{22} - a_{12}\,a_{21}) \geq 0$ since (1,2) is completely defined.
A (1,2,5)	Det A (1,2,5) = $d_{55}(d_{11}\,d_{22} - a_{12}\,a_{21}) \geq 0$ since (1,2) is completely defined.
A (1,3,4)	Det A (1,3,4) = $d_{11}\,d_{33}\,d_{44} \geq 0$
A (1,3,5)	Det A (1,3,5) = $d_{11}\,d_{33}\,d_{55} \geq 0$
A (1,4,5)	Det A (1,4,5) = $d_{11}\,d_{44}\,d_{55} \geq 0$
A (2,3,4)	Det A (2,3,4) = $d_{22}\,d_{33}\,d_{44} \geq 0$
A (2,3,5)	Det A (2,3,5) = $d_{22}\,d_{33}\,d_{55} \geq 0$
A (2,4,5)	Det A (2,4,5) = $d_{22}\,d_{44}\,d_{55} \geq 0$
A (3,4,5)	Det A (3,4,5) = $d_{33}\,d_{44}\,d_{55} \geq 0$

Source: Researcher (2024)

Table 4. 15

Determinants of 4×4 sub-matrices of " clique digraph, order 5, 3 arc."

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{33}\, d_{44}\, (d_{11}\, d_{22} - a_{12}\, a_{21}) \geq 0$ since (1,2) is completely defined.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{33}\, d_{55}\, (d_{11}\, d_{22} - a_{12}\, a_{21}) \geq 0$ since (1,2) is completely defined.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{33}\, d_{55}\, (d_{11}\, d_{22} - a_{12}\, a_{21}) \geq 0$ since (1,2) is completely defined.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}\, d_{33}\, d_{44}\, d_{55} \geq 0$
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$

Source: Researcher (2024)

Det A = $d_{44}\, d_{55}\, (d_{11}\, d_{22} - a_{12}\, a_{21}) \geq 0$. Since (1,2) is completely defined.

Det A (1,3) = $d_{11}\, d_{33} - a_{13}\, x_{31} \geq 0$. Since, $a_{35}\, x_{53} = 0$ after assigning $x_{31} = 0$.

Since all the determinants are non-negative then the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

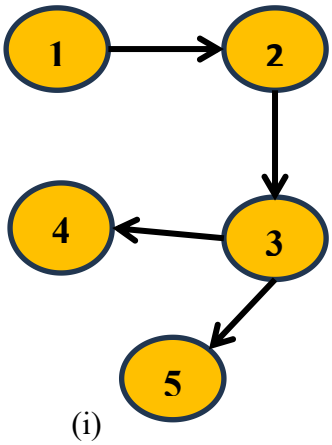
All other digraphs were solved in a similar manner and were found to have zero completion into a Wss P_0-matrix.

4.5 Digraphs of Order 5 with 4 Arcs

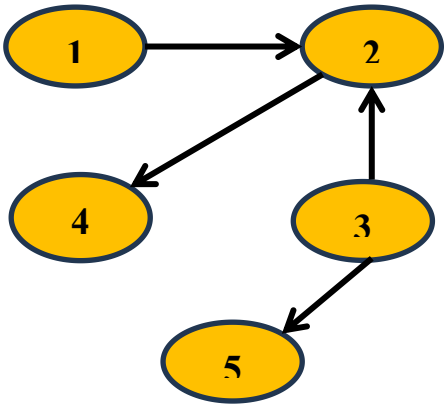
According to the technique developed by Harary *et al. (*2018), there are 61 non-isomorphic digraphs with 5 vertices and 4 arcs. These are few examples of digraphs with 5 vertices and 4 arcs.

Figure 4. 8

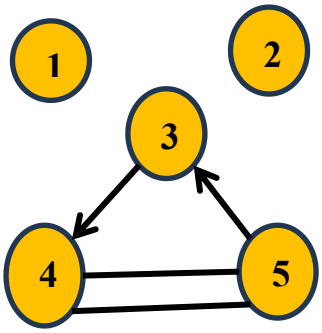
Digraphs with 5 vertices and 4 arcs.



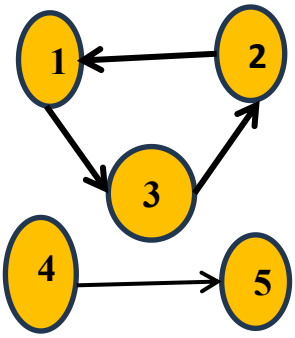
(i)



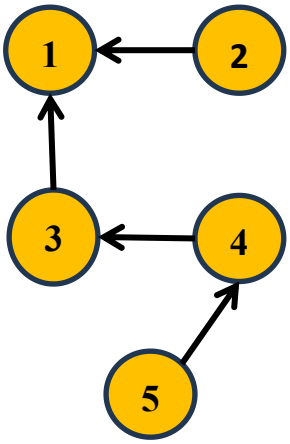
(ii)



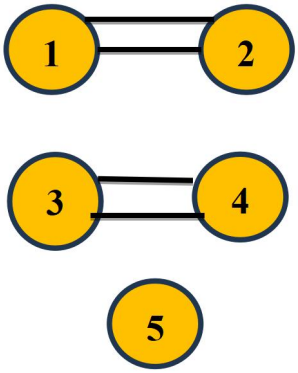
(iii)



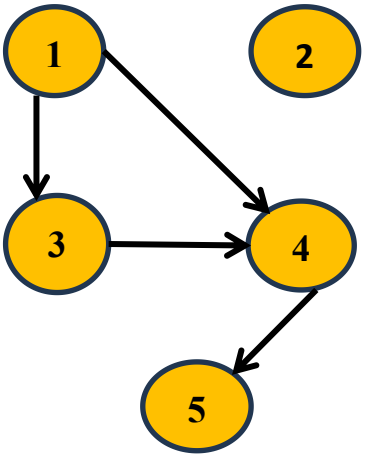
(iv)



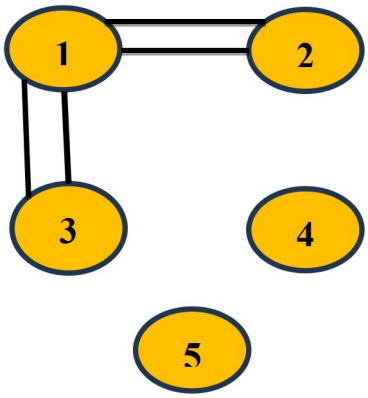
(v)



(vi)



(vii)



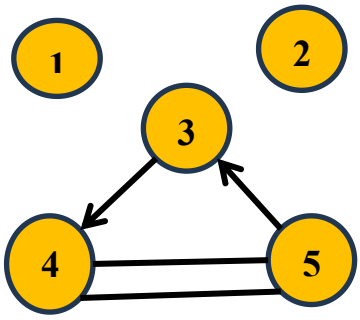
(viii)

Source: Researcher (2024)

Example 6

Figure 4. 9

Cyclic Digraph D of order 5 with 4 arcs



Source: Researcher (2024)

The matrix that partially outlines the above digraph is A=$\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & a_{53} & a_{54} & d_{55} \end{pmatrix}$

The other digraphs and their corresponding partial Wss P_0-matrices could be constructed and extracted respectively, in an analogous manner.

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

A (1, 2) = $\begin{bmatrix} d_{11} & x_{12} \\ x_{21} & d_{22} \end{bmatrix}$, A (1, 3) = $\begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$A (1, 4) = $\begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$,

A (1, 5) = $\begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, A (2, 3) = $\begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$A (2, 4) = $\begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$,

A (2, 5) = $\begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, A (3, 4) = $\begin{bmatrix} d_{33} & a_{34} \\ x_{43} & d_{44} \end{bmatrix}$

A (3, 5) = $\begin{bmatrix} d_{33} & x_{35} \\ a_{53} & d_{55} \end{bmatrix}$, A (4, 5) = $\begin{bmatrix} d_{44} & a_{45} \\ a_{54} & d_{55} \end{bmatrix}$,A (1,2,3) = $\begin{bmatrix} d_{11} & x_{12} & x_{13} \\ x_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}$,

A (1,2,4) = $\begin{bmatrix} d_{11} & x_{12} & x_{14} \\ x_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}$, A (1, 2,5)= $\begin{bmatrix} d_{11} & x_{12} & x_{15} \\ x_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$

A (1, 3, 4) = $\begin{bmatrix} d_{11} & x_{13} & x_{14} \\ x_{31} & d_{33} & a_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix}$, A (1, 3, 5) = $\begin{bmatrix} d_{11} & x_{13} & x_{15} \\ x_{31} & d_{33} & x_{35} \\ x_{51} & a_{53} & d_{55} \end{bmatrix}$,

A (1, 4,5) = $\begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & a_{45} \\ x_{51} & a_{54} & d_{55} \end{bmatrix}$,A (2, 3, 4) = $\begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & a_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}$,

A (2, 3, 5) = $\begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & a_{53} & d_{55} \end{bmatrix}$, A (2, 4, 5) = $\begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & a_{45} \\ x_{52} & a_{54} & d_{55} \end{pmatrix}$

A (3, 4, 5) = $\begin{pmatrix} d_{33} & a_{34} & x_{35} \\ x_{43} & d_{44} & a_{45} \\ a_{53} & a_{54} & d_{55} \end{pmatrix}$, A (1, 2, 3, 4) = $\begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & a_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix}$,

$$\text{A } (1, 2, 3, 5) = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ x_{51} & x_{52} & a_{53} & d_{55} \end{pmatrix}, \text{A } (1, 2, 4, 5) = \begin{pmatrix} d_{11} & x_{12} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & a_{45} \\ x_{51} & x_{52} & a_{54} & d_{55} \end{pmatrix},$$

$$\text{A } (1, 3, 4, 5) = \begin{pmatrix} d_{11} & x_{13} & x_{14} & x_{14} \\ x_{31} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{43} & d_{44} & a_{45} \\ x_{51} & a_{53} & a_{54} & d_{55} \end{pmatrix}, \text{A } (2, 3, 4, 5) = \begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & a_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & a_{45} \\ x_{52} & a_{53} & a_{54} & d_{55} \end{pmatrix}$$

Determinants of each sub-matrix obtained were obtained as indicated:

Det (1,2) = $d_{11}\, d_{22} - x_{12}\, x_{21}$

Det A (1,3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - a_{34}\, x_{43}$

Det A (3,5) = $d_{33}\, d_{55} - x_{35}\, a_{53}$

Det A (4,5) = $d_{44}\, d_{55} - a_{45}\, a_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32}) - x_{12}\,(x_{21}\, d_{33} - x_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22}\, x_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - x_{12}\,(x_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(x_{21}\, x_{42} - d_{22}\, x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52}) - x_{12}\,(x_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(x_{21}\, x_{52} - d_{22}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}\, d_{44} - a_{34}\, a_{43}) - x_{13}\,(x_{31}\, d_{44} - a_{34}\, x_{41}) + x_{14}\,(x_{31}\, x_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}\, d_{55} - x_{35}\, a_{53}) - x_{13}\,(x_{31}\, d_{55} - x_{35}\, x_{51}) + x_{15}\,(x_{31}\, a_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}\, d_{55} - a_{45}\, a_{54}) - x_{14}\,(x_{41}\, d_{55} - a_{45}\, x_{51}) + x_{15}\,(x_{41}\, a_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}\, d_{44} - a_{34}\, x_{43}) - x_{23}\,(x_{32}\, d_{44} - a_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$

$$\text{Det A } (2,3,5) = d_{22}\,(d_{33}\,d_{55} - x_{35}\,a_{53}) - x_{23}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) + x_{25}\,(x_{32}\,a_{53} - d_{33}\,x_{52})$$

$$\text{Det A } (2,4,5) = d_{22}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{24}\,(x_{42}\,d_{55} - a_{45}\,x_{52}) + x_{25}\,(x_{42}\,a_{54} - d_{44}\,x_{52})$$

$$\text{Det A } (3,4,5) = d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(x_{43}\,d_{55} - a_{45}\,a_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,a_{53})$$

$$\begin{aligned}\text{Det A } (1,2,3,4) = {} & d_{11}\,[d_{22}\,(d_{33}\,d_{44} - a_{34}\,x_{43}) - x_{23}\,(d_{44}\,x_{32} - a_{34}\,x_{42}) + x_{24}\,(x_{32}\,x_{43} - d_{33}\,x_{42})] \\ & - x_{12}\,[x_{21}\,(d_{33}\,d_{44} - a_{34}\,x_{43}) - x_{23}\,(d_{44}\,x_{31} - a_{34}\,x_{41}) + x_{24}\,(x_{31}\,x_{43} - d_{33}\,x_{41})] \\ & + x_{13}\,[x_{21}\,(x_{32}\,d_{44} - a_{34}\,x_{42}) - d_{22}\,(d_{44}\,x_{31} - a_{34}\,x_{41}) + x_{24}\,(x_{31}\,x_{42} - x_{32}\,x_{41})] \\ & - x_{14}\,[x_{21}\,(x_{32}\,x_{43} - d_{33}\,x_{42}) - d_{22}\,(x_{43}\,x_{31} - d_{33}\,x_{41}) + x_{23}\,(x_{31}\,x_{42} - x_{32}\,x_{41})]\end{aligned}$$

$$\begin{aligned}\text{Det A } (1,2,3,5) = {} & d_{11}\,[d_{22}\,(d_{33}\,d_{55} - x_{35}\,a_{53}) - x_{23}\,(d_{55}\,x_{32} - x_{35}\,x_{52}) + x_{25}\,(x_{32}\,a_{53} - d_{33}\,x_{52})] \\ & - x_{12}\,[x_{21}\,(d_{33}\,d_{55} - x_{35}\,a_{53}) - x_{23}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,a_{53} - d_{33}\,x_{51})] \\ & + x_{13}\,[x_{21}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) - d_{22}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,x_{52} - x_{32}\,x_{51})] \\ & - x_{15}\,[x_{21}\,(x_{32}\,a_{53} - d_{33}\,x_{52}) - d_{22}\,(a_{53}\,x_{31} - d_{33}\,x_{51}) + x_{23}\,(x_{31}\,x_{52} - x_{32}\,x_{51})]\end{aligned}$$

$$\begin{aligned}\text{Det A } (1,2,4,5) = {} & d_{11}\,[d_{22}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{24}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{25}\,(x_{42}\,a_{54} - d_{44}\,x_{52})] \\ & - x_{12}\,[x_{21}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{24}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{25}\,(x_{41}\,a_{54} - d_{44}\,x_{51})] \\ & + x_{14}\,[x_{21}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) - d_{22}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,x_{51})] \\ & - x_{15}\,[x_{21}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) - d_{22}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{24}\,(x_{41}\,x_{52} - x_{51}\,x_{42})]\end{aligned}$$

$$\begin{aligned}\text{Det A } (1,3,4,5) = {} & d_{11}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{43} - a_{45}\,a_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,x_{53})] \\ & - x_{13}\,[x_{31}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51})] \\ & + x_{14}\,[x_{31}\,(d_{55}\,x_{43} - a_{45}\,a_{53}) - d_{33}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{53} - x_{43}\,x_{51})] \\ & - x_{15}\,[x_{31}\,(x_{43}\,a_{54} - d_{44}\,a_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,a_{53} - x_{51}\,x_{43})]\end{aligned}$$

$$\begin{aligned}\text{Det A } (2,3,4,5) = {} & d_{22}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{43} - a_{45}\,a_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,a_{53})] \\ & - x_{23}\,[x_{32}\,(x_{43}\,d_{55} - a_{45}\,a_{53}) - a_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{53} - x_{43}\,x_{52})] \\ & + x_{24}\,[x_{32}\,(d_{55}\,x_{43} - a_{45}\,a_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{53} - x_{43}\,x_{52})] \\ & - x_{23}\,[x_{32}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52})]\end{aligned}$$

$$\begin{aligned}
\text{Det A} = d_{11}\,[&d_{22}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{43} - a_{45}\,a_{53}) + x_{35}\,(x_{43}a_{54} - d_{44}\,a_{53}))\\
&- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52}))\\
&+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - a_{45}\,a_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{53} - x_{43}\,x_{52}))\\
&- x_{25}\,(x_{32}\,(a_{43}\,a_{54} - d_{44}\,a_{53}) - d_{33}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,a_{53} - x_{43}\,x_{52}))]\\
- x_{12}\,[&x_{21}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{43} - a_{45}\,a_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,a_{53}))\\
&+ x_{23}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51}))\\
&- x_{24}\,(x_{31}\,(x_{43}\,d_{55} - a_{45}\,a_{53}) - d_{33}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{53} - x_{43}\,x_{51}))\\
&+ x_{25}\,(x_{31}\,(x_{43}\,a_{54} - d_{44}\,a_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,a_{53} - x_{43}\,x_{51}))]\\
+ x_{13}\,[&x_{21}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52}))\\
&- d_{22}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - a_{34}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51}))\\
&+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - a_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))\\
&- x_{25}\,(x_{31}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]\\
- x_{14}\,[&x_{21}\,(x_{32}(d_{55}\,x_{43} - a_{45}\,a_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{53} - x_{43}\,x_{52}))\\
&+ d_{22}\,(x_{31}\,(d_{55}\,x_{43} - a_{45}\,a_{53}) - d_{33}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{53} - x_{43}\,x_{51}))\\
&- x_{23}\,(x_{31}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))\\
&+ \ x_{25}\,(x_{31}\,(a_{53}\,x_{42} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,a_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]\\
+ x_{15}\,[&x_{21}\,(x_{32}(x_{43}\,a_{54} - d_{44}\,a_{53}) - d_{33}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,a_{53} - x_{43}\,x_{52}))\\
&- d_{22}\,(x_{31}\,(a_{54}\,x_{43} - d_{44}\,a_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,a_{53} - x_{43}\,x_{51}))\\
&+ x_{23}\,(x_{31}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))\\
&- x_{24}\,(x_{31}\,(x_{42}\,a_{53} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,a_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]
\end{aligned}$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}=\ x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 16

Determinants of 2×2 sub-matrices of " cyclic digraph, order 5, 4 arc."

Principal sub-matrix	**Principal minor**
A (1,2)	Det A (1,2) = $d_{11}\, d_{22} \geq 0$
A (1,3)	Det A (1,3) = $d_{11}\, d_{33} \geq 0$
A (1,4)	Det A (1,4) = $d_{11}\, d_{44} \geq 0$
A (1,5)	Det A (1,5) = $d_{11}\, d_{55} \geq 0$
A (2,3)	Det A (2,3) = $d_{22}\, d_{33} \geq 0$
A (2,4)	Det A (2,4) = $d_{22}\, d_{44} \geq 0$
A (2,5)	Det A (2,5) = $d_{22}\, d_{55} \geq 0$
A (3,4)	Det A (3,4) = $d_{33}\, d_{44} \geq 0$
A (3,5)	Det A (3,5) = $d_{33}\, d_{55} \geq 0$
A (4,5)	Det A (4,5) = $(d_{44}\, d_{55} - a_{45}\, a_{54}) \geq 0$ (since (4,5) is completely defined.

Source: Researcher (2024)

Table 4. 17

Determinants of 3×3 sub-matrices of " cyclic digraph, order 5, 4 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = $d_{11}\, d_{22}\, d_{33} \geq 0$
A (1,2,4)	Det A (1,2,4) = $d_{11}\, d_{22}\, d_{44} \geq 0$
A (1,2,5)	Det A (1,2,5) = $d_{11}\, d_{22}\, d_{55} \geq 0$
A (1,3,4)	Det A (1,3,4) = $d_{11}\, d_{33}\, d_{44} \geq 0$
A (1,3,5)	Det A (1,3,5) = $d_{11}\, d_{33}\, d_{55} \geq 0$
A (1,4,5)	Det A (1,4,5) = $d_{11}\, (d_{44}\, d_{55} - a_{45}\, a_{54}) \geq 0$ (since (4,5) is completely defined).
A (2,3,4)	Det A (2,3,4) = $d_{22}\, d_{33}\, d_{44} \geq 0$
A (2,3,5)	Det A (2,3,5) = $d_{22}\, d_{33}\, d_{55} \geq 0$
A (2,4,5)	Det A (2,4,5) = $d_{22}\, (d_{44}\, d_{55} - a_{45}\, a_{54}) \geq 0$ (since (4,5) is completely defined).
A (3,4,5)	Det A (3,4,5) = $d_{33}\, (d_{33}\, d_{44} - a_{45}\, a_{54}) + a_{34}\, a_{45}\, a_{53} \geq 0$

Source: Researcher (2024)

Table 4. 18

Determinants of 4×4 sub-matrices of " cyclic digraph, order 5, 4 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}\, d_{22}\, d_{33}\, d_{44} \geq 0$
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}\, d_{22}\, d_{33}\, d_{55} \geq 0$
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}\, d_{22}\, (d_{44}\, d_{55} - a_{45}\, a_{54}) \geq 0$ (since (4,5) is completely defined).
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}\, d_{33}\, (d_{44}\, d_{55} - a_{45}\, a_{54}) + d_{11} a_{34} a_{45} a_{53} \geq 0$
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\, d_{33}\, (d_{44}\, d_{55} - a_{45}\, a_{54}) + d_{33}\, a_{34}\, a_{45}\, a_{53} \geq 0$.

Source: Researcher (2024)

Det A = $d_{11}\, d_{22}\, d_{33}\, (d_{44}\, d_{55} - a_{45}\, a_{54}) + d_{11}\, d_{22}\, a_{34}\, a_{45}\, a_{53} \geq 0$.

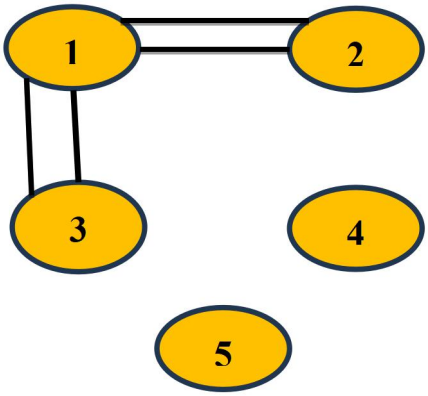
Det A (3,4) = $d_{33}\ d_{44} - a_{34}\ x_{43} \geq 0$. since, $a_{34}\ x_{43} = 0$ after assigning $x_{31} = 0$. Since all the determinants are non-negative then the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

Case 2

Example 7

Figure 4. 10

A digraph with clique sub-digraph



Source: Researcher (2024)

The partial matrix that specifies the above digraph A is = $\begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ a_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

A (1, 2) = $\begin{bmatrix} d_{11} & a_{12} \\ a_{21} & d_{22} \end{bmatrix}$, A (1, 3) = $\begin{bmatrix} d_{11} & a_{13} \\ a_{31} & d_{33} \end{bmatrix}$

A (1, 4) = $\begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$, A (1, 5) = $\begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, A (2, 3) = $\begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$

A (2, 4) = $\begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$, A (2, 5) = $\begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, A (3, 4) = $\begin{bmatrix} d_{33} & x_{34} \\ x_{43} & d_{44} \end{bmatrix}$

A (3, 5) = $\begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}$, A (4, 5) = $\begin{bmatrix} d_{44} & x_{45} \\ x_{54} & d_{55} \end{bmatrix}$

A (1, 2, 3) = $\begin{bmatrix} d_{11} & a_{12} & a_{13} \\ a_{21} & d_{22} & x_{23} \\ a_{31} & x_{32} & d_{33} \end{bmatrix}$, A (1, 2, 4) = $\begin{bmatrix} d_{11} & a_{12} & x_{14} \\ a_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}$,

A (1, 2, 5) = $\begin{bmatrix} d_{11} & a_{12} & x_{15} \\ a_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$A (1, 3, 4) = $\begin{bmatrix} d_{11} & a_{13} & x_{14} \\ a_{31} & d_{33} & x_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix}$,

A (1, 3, 5) = $\begin{bmatrix} d_{11} & a_{13} & x_{15} \\ a_{31} & d_{33} & x_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}$, A (1, 4, 5) = $\begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & x_{54} & d_{55} \end{bmatrix}$

A (2, 3, 4) = $\begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & x_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}$, A (2, 3, 5) = $\begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}$,

A (2, 4, 5) = $\begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$, A (3, 4, 5) = $\begin{pmatrix} d_{33} & x_{34} & x_{35} \\ x_{43} & d_{44} & x_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix}$,

A (1, 2, 3, 4) = $\begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} \\ a_{21} & d_{22} & x_{23} & x_{24} \\ a_{31} & x_{32} & d_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix}$, A (1, 2, 3, 5) = $\begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{25} \\ a_{31} & x_{32} & d_{33} & x_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}$,

A (1, 2, 4, 5) = $\begin{pmatrix} d_{11} & a_{12} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$, A (1, 3, 4, 5) = $\begin{pmatrix} d_{11} & a_{13} & x_{14} & x_{15} \\ a_{31} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{53} & x_{54} & d_{55} \end{pmatrix}$,

A (2, 3, 4, 5) = $\begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1,2) = $d_{11}\, d_{22} - a_{12}\, a_{21}$

Det A (1,3) = $d_{11}\, d_{33} - a_{13}\, a_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - x_{34}\, x_{43}$

Det A (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$

Det A (4,5) = $d_{44}\, d_{55} - x_{45}\, x_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32)} - a_{12}\,(a_{21}\, d_{33} - x_{23}\, x_{31}) + a_{13}\,(a_{21}\, x_{32} - d_{22}\, a_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42)} - a_{12}\,(a_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(a_{21}\, x_{42} - d_{22}\, x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52)} - a_{12}\,(a_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(a_{21}\, x_{52} - d_{22}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}\, d_{44} - x_{34}\, x_{43)} - a_{13}\,(a_{31}\, d_{44} - x_{34}\, x_{41}) + x_{14}\,(a_{31}\, x_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}\, d_{55} - x_{35}\, x_{53)} - a_{13}\,(x_{31}\, d_{55} - x_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{14}\,(x_{41}\, d_{55} - x_{45}\, x_{51}) + x_{15}\,(x_{41}\, x_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$

Det A (2,3,5) = $d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$

Det A (2,4,5) = $d_{22}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(x_{42}\, d_{55} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})$

Det A (3,4,5) = $d_{33}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{34}\,(x_{43}\, d_{55} - x_{45}\, x_{53}) + x_{35}\,(x_{43}\, x_{54} - d_{44}\, x_{53})$

Det A (1,2,3,4) = $d_{11}\,[d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})]$

$- a_{12}\,[a_{21}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, x_{41})]$

$+ a_{13}\,[a_{21}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$

$- x_{14}\,[a_{21}\,(x_{32}\, x_{43} - d_{33}\, x_{42}) - d_{22}\,(x_{43}\, x_{31} - d_{33}\, x_{41}) + x_{23}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$

Det A (1,2,3,5) = $d_{11}\,[d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{32} - x_{35}\, x_{52}) + x_{25}(x_{32}x_{53} - d_{33}\, x_{52})]$

$$- a_{12}\,[a_{21}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - x_{23}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,x_{53} - d_{33}\,x_{51})]$$

$$+ a_{13}\,[a_{21}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) - d_{22}\,(d_{55}\,x_{31} - x_{35}\,x_{51}) + x_{25}\,(x_{31}\,x_{52} - x_{32}\,x_{51})]$$

$$- x_{15}\,[a_{21}\,(x_{32}\,x_{53} - d_{33}\,x_{52}) - d_{22}\,(x_{53}\,x_{31} - d_{33}\,x_{51}) + x_{23}\,(x_{31}\,x_{52} - x_{32}\,x_{51})]$$

$$\text{Det A } (1,2,4,5) = d_{11}\,[d_{22}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{24}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{25}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$- a_{12}\,[a_{21}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{24}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[a_{21}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - d_{22}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,x_{51})]$$

$$- x_{15}\,[a_{21}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - d_{22}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{24}\,(x_{41}\,x_{52} - x_{51}\,x_{42})]$$

$$\text{Det A } (1,3,4,5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- a_{13}\,[x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{51}\,x_{43})]$$

$$\text{Det A } (2,3,4,5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{23}\,[x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- \; x_{25}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))]$$

$$- a_{12}\,[a_{21}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$+ x_{23}\,(a_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$- x_{24}\,(a_{31}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ x_{25}\,(a_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))]$$

$$+ a_{13}\,[a_{21}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$- d_{22}\,(a_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(a_{31}\,(x_{42}\,d_{55} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{25}\,(a_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_4\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$- x_{14}\,[a_{21}\,(x_{32}(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$+ d_{22}\,(a_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$- x_{23}\,(a_{31}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$x_{25}\,(a_{31}\,(x_{53}\,x_{42} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$+ x_{15}\,[a_{21}\,(x_{32}(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(a_{31}\,(x_{54}\,x_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ x_{23}\,(a_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{24}\,(a_{31}\,(x_{42}\,x_{53} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}= x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 19

Determinants of 2×2 sub-matrices of " clique digraph, order 5, 4 arc."

Principal sub-matrix	**Principal minor**
A (1,2)	Det A (1,2) = $(d_{11}\, d_{22} - a_{12}\, a_{21}) \geq 0$ (since (1,2) is completely defined).
A (1,3)	Det A (1,3) = $(d_{11}\, d_{33} - a_{13}\, a_{31}) \geq 0$ (since (1,3) is completely defined).
A (1,4)	Det A (1,4) = $d_{11}\, d_{44} \geq 0$
A (1,5)	Det A (1,5) = $d_{11}\, d_{55} \geq 0$
A (2,3)	Det A (2,3) = $d_{22}\, d_{33} \geq 0$
A (2,4)	Det A (2,4) = $d_{22}\, d_{44} \geq 0$
A (2,5)	Det A (2,5) = $d_{22}\, d_{55} \geq 0$
A (3,4)	Det A (3,4) = $d_{33}\, d_{44} \geq 0$
A (3,5)	Det A (3,5) = $d_{33}\, d_{55} \geq 0$
A (4,5)	Det A (4,5) = $d_{44}\, d_{55} \geq 0$

Source: Researcher (2024)

Table 4. 20

Determinants of 3×3 sub-matrices of " clique digraph, order 5, 4 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = d_3 (d_{11} d_{22} — a_{12} a_{21}) — a_{13} a_{31} d_{22}
A (1,2,4)	Det A (1,2,4) = d_{44} (d_{11} d_{22} – a_{12} a_{21}) ≥ 0 (since (1,2) is completely defined).
A (1,2,5)	Det A (1,2,5) = d_{55} (d_{11} d_{22} – a_{12} a_{21}) ≥ 0 (since (1,2) is fully specified).
A (1,3,4)	Det A (1,3,4) = d_{44} (d_{11} d_{33} – a_{13} a_{31}) ≥ 0 (since (1,3) is completely defined).
A (1,3,5)	Det A (1,3,5) = d_{55} (d_{11} d_{33} – a_{13} a_{31}) ≥ 0 (since (1,3) is completely defined).
A (1,4,5)	Det A (1,4,5) = d_{11} d_{44} d_{55} ≥ 0
A (2,3,4)	Det A (2,3,4) = d_{22} d_{33} d_{44} ≥ 0
A (2,3,5)	Det A (2,3,5) = d_{22} d_{33} d_{55} ≥ 0
A (2,4,5)	Det A (2,4,5) = d_{22} d_{44} d_{55} ≥ 0
A (3,4,5)	Det A (3,4,5) = d_{33} d_{44} d_{55} ≥ 0

Source: Researcher (2024)

Table 4. 21

Determinants of 4×4 sub-matrices of " clique digraph, order 5, 4 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3,4)	Det A (1,2,3,4) = $d_{33}\, d_{44}\, (d_{11}\, d_{22} - a_{12}\, a_{21}) - a_{13}\, a_{31}\, d_{22}\, d_{44}$
A (1,2,3,5)	Det A (1,2,3,5) = $d_{33}\, d_{55}\, (d_{11}\, d_{22} - a_{12}\, a_{21}) - a_{13}\, a_{31}\, d_{22}\, d_{55}$
A (1,2,4,5)	Det A (1,2,4,5) = $d_{44}\, d_{55}\, (d_{11}\, d_{22} - a_{12}\, a_{21}) \geq 0$ (since (1,2) is completely defined).
A (1,3,4,5)	Det A (1,3,4,5) = $d_{44}\, d_{55}\, (d_{11}\, d_{33} - a_{13}\, a_{31}) \geq 0$ (since (1,3) is completely defined).
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$

Det A = $d_{11}\, d_{22}\, d_{33}\, d_{44}\, d_{55} - a_{12}\, a_{21}\, d_{33}\, d_{44}\, d_{55} - a_{13}\, a_{31}\, d_{22}\, d_{44}\, d_{55}$

Since not all the determinants are non-negative, the partial matrix cannot be completed into a Wss P_0-matrix. Therefore, it was found not to admit zero completion into a Wss P_0-matrix.

Counter example

Let the partial matrix specifying the sub digraph (1,2,3) be

M = $\begin{bmatrix} 2 & -2 & -2 \\ -1 & 2 & x_{23} \\ -2 & x_{32} & 2 \end{bmatrix}$. After substituting the unspecified entries with zero i.e. $x_{23} = x_{32} = 0$.

Then

|M| = $d_{11}\, d_{22}\, d_{33} - a_{12}\, a_{21}\, d_{33} - a_{13}\, d_{22} d_{33}$

= 8 – 4 –8 = – 4 < 0. Hence M (1,2,3) has no completion.

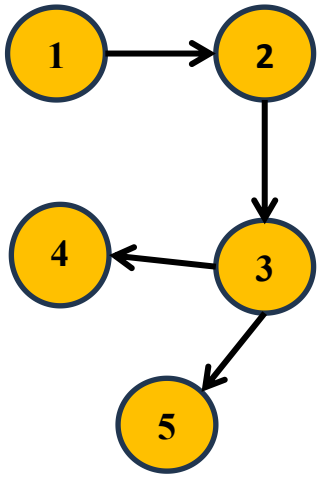
Case 3

Example 8:

Assume a digraph D = {(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (3,5), (4,4), (5,5)} with 5 vertices and 4 arcs given by:

Figure 4. 11

Acyclic digraph D of order 5 and 4 arcs



Source: Researcher (2024)

The matrix that partially outlines above digraph is $A = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

$A(1, 2) = \begin{bmatrix} d_{11} & a_{12} \\ x_{21} & d_{22} \end{bmatrix}$, $A(1, 3) = \begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$

$A(1, 4) = \begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$, $A(1, 5) = \begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, $A(2, 3) = \begin{bmatrix} d_{22} & a_{23} \\ x_{32} & d_{33} \end{bmatrix}$

$A(2, 4) = \begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$, $A(2, 5) = \begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, $A(3, 4) = \begin{bmatrix} d_{33} & a_{34} \\ x_{43} & d_{44} \end{bmatrix}$

A (3, 5) = $\begin{bmatrix} d_{33} & a_{35} \\ x_{53} & d_{55} \end{bmatrix}$, A (4, 5) = $\begin{bmatrix} d_{44} & x_{45} \\ x_{54} & d_{55} \end{bmatrix}$,

A(1,2,3) = $\begin{bmatrix} d_{11} & a_{12} & x_{13} \\ x_{21} & d_{22} & a_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}$,A(1,2,4) = $\begin{bmatrix} d_{11} & a_{12} & x_{14} \\ x_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}$, A(1, 2, 5)= $\begin{bmatrix} d_{11} & a_{12} & x_{15} \\ x_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$

A(1,3,4) = $\begin{bmatrix} d_{11} & x_{13} & x_{14} \\ x_{31} & d_{33} & a_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix}$, A(1,3,5)= $\begin{bmatrix} d_{11} & x_{13} & x_{15} \\ x_{31} & d_{33} & a_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}$, A(1, 4, 5) = $\begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & x_{54} & d_{55} \end{bmatrix}$

A (2,3,4)= $\begin{bmatrix} d_{22} & a_{23} & x_{24} \\ x_{32} & d_{33} & a_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}$, A(2,3,5)= $\begin{bmatrix} d_{22} & a_{23} & x_{25} \\ x_{32} & d_{33} & a_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}$, A (2,4,5) = $\begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$

A (3, 4, 5) = $\begin{pmatrix} d_{33} & a_{34} & a_{35} \\ x_{43} & d_{44} & x_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix}$, A (1, 2, 3, 4) = $\begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} \\ x_{21} & d_{22} & a_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & a_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix}$,

A (1, 2, 3, 5) = $\begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}$, A (1, 2, 4, 5) = $\begin{pmatrix} d_{11} & a_{12} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$

A (1, 3, 4, 5) = $\begin{pmatrix} d_{11} & x_{13} & x_{14} & x_{15} \\ x_{31} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$, A (2, 3, 4, 5) = $\begin{pmatrix} d_{22} & a_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & a_{34} & a_{35} \\ x_{42} & x_{43} & d_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1, 2) = $d_{11}\, d_{22} - a_{12}\, x_{21}$

Det A (1, 3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$

Det A (1, 4) = $d_{11}\, d_{44} - x_{14}\, x_{41}$

Det A (1, 5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2, 3) = $d_{22}\, d_{33} - a_{23}\, x_{32}$

Det A (2, 4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2, 5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

$$\text{Det A } (3, 4) = d_{33}\, d_{44} - a_{34}\, x_{43}$$

$$\text{Det A } (3, 5) = d_{33}\, d_{55} - a_{35}\, x_{53}$$

$$\text{Det A } (4, 5) = d_{44}\, d_{55} - x_{45}\, x_{54}$$

$$\text{Det A } (1, 2, 3) = d_{11}\,(d_{22}\, d_{33} - a_{23}\, x_{32}) - a_{12}\,(x_{21}\, d_{33} - a_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22}\, x_{31})$$

$$\text{Det A } (1, 2, 4) = d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - a_{12}\,(x_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(x_{21}\, x_{42} - d_{22}\, x_{41})$$

$$\text{Det A } (1, 2, 5) = d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52}) - a_{12}\,(x_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(x_{21}\, x_{52} - d_{22}\, x_{51})$$

$$\text{Det A } (1, 3, 4) = d_{11}\,(d_{33}\, d_{44} - a_{34}\, x_{43}) - x_{13}\,(x_{31}\, d_{44} - a_{34}\, x_{41}) + x_{14}\,(x_{31}\, x_{43} - d_{33}\, x_{41})$$

$$\text{Det A } (1, 3, 5) = d_{11}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - x_{13}\,(x_{31}\, d_{55} - a_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$$

$$\text{Det A } (1, 4, 5) = d_{11}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{14}\,(x_{41}\, d_{55} - x_{45}\, x_{51}) + x_{15}\,(x_{41}\, x_{54} - d_{44}\, x_{51})$$

$$\text{Det A } (2, 3, 4) = d_{22}\,(d_{33}\, d_{44} - a_{34}\, x_{43}) - a_{23}\,(x_{32}\, d_{44} - a_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$$

$$\text{Det A } (2, 3, 5) = d_{22}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - a_{23}\,(x_{32}\, d_{55} - a_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$$

$$\text{Det A } (2, 4, 5) = d_{22}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(x_{42}\, d_{55} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52}).$$

$$\text{Det A } (3, 4, 5) = d_{33}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - a_{34}\,(x_{43}\, d_{55} - x_{45}\, x_{53}) + a_{35}\,(x_{43}\, x_{54} - d_{44}\, x_{53}).$$

$$\begin{aligned}\text{Det A } (1, 2, 3, 4) = {} & d_{11}\,[d_{22}\,(d_{33}\, d_{44} - a_{34}\, x_{43}) - a_{23}\,(d_{44}\, x_{32} - a_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})] \\ & - a_{12}\,[x_{21}\,(d_{33}\, d_{44} - a_{34}\, x_{43}) - a_{23}\,(d_{44}\, x_{31} - a_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, x_{41})] \\ & + x_{13}\,[x_{21}\,(x_{32}\, d_{44} - a_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - a_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})] \\ & - x_{14}\,[x_{21}\,(x_{32}\, x_{43} - d_{33}\, x_{42}) - d_{22}(x_{43}\, x_{31} - d_{33}\, x_{41}) + a_{23}\,(x_{31}\, x_{42} - x_{32}\, x_{41})].\end{aligned}$$

$$\begin{aligned}\text{Det A } (1, 2, 3, 5) = {} & d_{11}\,[d_{22}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - a_{23}\,(d_{55}\, x_{32} - a_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})] \\ & - a_{12}\,[x_{21}\,(d_{33}\, d_{55} - a_{35}\, x_{53}) - a_{23}\,(d_{55}\, x_{31} - a_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{53} - d_{33}\, x_{51})] \\ & + x_{13}\,[x_{21}\,(x_{32}\, d_{55} - a_{35}\, x_{52}) - d_{22}\,(d_{55}\, x_{31} - a_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{52} - x_{32}\, x_{51})] \\ & - x_{15}\,[x_{21}\,(x_{32}\, x_{53} - d_{33}\, x_{52}) - d_{22}\,(x_{53}\, x_{31} - d_{33}\, x_{51}) + a_{23}\,(x_{31}\, x_{52} - x_{32}\, x_{51})].\end{aligned}$$

$$\begin{aligned}\text{Det A } (1, 2, 4, 5) = {} & d_{11}\,[d_{22}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(d_5\, x_{42} - x_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52})] \\ & - a_{12}\,[x_{21}\,(d_{44}\, d_{55} - x_{45}\, x_{54}) - x_{24}\,(d_5\, x_{41} - x_{45}\, x_{51}) + x_{25}\,(x_{41}\, x_{54} - d_{44}\, x_{51})]\end{aligned}$$

$$+ x_{14}\,[x_{21}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - d_{22}\,(d_5\,x_{41} - x_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,x_{51})]$$

$$- x_{15}\,[x_{21}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - d_2\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + x_{24}\,(x_{41}\,x_{52} - x_{51}\,x_{42})].$$

Det A (1, 3, 4, 5) $= d_{11}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_5\,x_{43} - x_{45}\,x_{53}) + a_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$

$$- x_{13}\,[x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_5\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_5\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{51}\,x_{43})].$$

Det A (2, 3, 4, 5) $= d_{22}\,[d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + a_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$

$$- a_{23}\,[x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{25}\,[x_{32}\,(x_{43}\,x_{54} - d_4\,x_{53}) - d_3\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52})].$$

Det A $= d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + a_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$

$$- a_{23}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- x_{25}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))]$$

$$- a_{12}\,[x_{21}\,(d_{33}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) + a_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$-x_{23}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{43}\,d_{55} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{44}\,d_{55} - x_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$-x_{25}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_4\,x_{51}) + a_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))] -$$

$$- x_{14}\,[x_{21}\,(x_{32}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) + a_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{55}\,x_{43} - x_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ a_{23}\,(x_{31}\,(d_{55}\,x_{42} - x_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - x_{45}\,x_{51}) + a_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{53}\,x_{42} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$+ x_{15}\,[x_{21}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(x_{54}\,x_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ a_{23}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{24}\,(x_{31}\,(x_{42}\,x_{53} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,x_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))].$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}= x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 22

Determinants of 2×2 sub-matrices of " acyclic digraph, order 5, 4 arc."

Principal sub-matrix	**Principal minor**
A (1, 2)	Det A (1, 2) = $d_{11}\,d_{22} \geq 0$.
A (1, 3)	Det A (1, 3) = $d_{11}\,d_{33} \geq 0$.
A (1, 4)	Det A (1, 4) = $d_{11}\,d_{44} \geq 0$
A (1, 5)	Det A (1, 5) = $d_{11}\,d_{55} \geq 0$
A (2, 3)	Det A (2, 3) = $d_{22}\,d_{33} \geq 0$
A (2, 4)	Det A (2, 4) = $d_{22}\,d_{44} \geq 0$
A (2, 5)	Det A (2, 5) = $d_{22}\,d_{55} \geq 0$
A (3, 4)	Det A (3 ,4) = $d_{33}\,d_{44} \geq 0$
A (3, 5)	Det A (3, 5) = $d_{33}\,d_{55} \geq 0$
A (4, 5)	Det A (4, 5) = $d_{44}\,d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 23

Determinants of 3×3 sub-matrices of " acyclic digraph, order 5, 4 arc."

Principal sub-matrix	Principal minor
A (1, 2, 3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$.
A (1, 2, 4)	Det A (1, 2, 4) = $d_{11}\, d_{22}\, d_{44} \geq 0$.
A (1, 2, 5)	Det A (1, 2, 5) = $d_{11}\, d_{22}\, d_{55} \geq 0$.
A (1, 3, 4)	Det A (1, 3, 4) = $d_{11}\, d_{33}\, d_{44} \geq 0$
A (1, 3, 5)	Det A (1, 3, 5) = $d_{11}\, d_{33}\, d_{55} \geq 0$
A (1, 4, 5)	Det A (1, 4, 5) = $d_{11}\, d_{44}\, d_{55} \geq 0$
A (2, 3, 4)	Det A (2, 3, 4) = $d_{22}\, d_{33}\, d_{44} \geq 0$
A (2, 3, 5)	Det A (2, 3, 5) = $d_{22}\, d_{33}\, d_{55} \geq 0$
A (2, 4, 5)	Det A (2, 4, 5) = $d_{22}\, d_{44}\, d_{55} \geq 0$
A (3, 4, 5)	Det A (3, 4, 5) = $d_{33}\, d_{44}\, d_{55} \geq 0$

Source: Researcher (2024)

Table 4. 24

Determinants of 4×4 sub-matrices of " acyclic digraph, order 5, 4 arc."

Principal sub-matrix	Principal minor
A (1, 2, 3, 4)	Det A (1, 2, 3, 4) = $d_{11}\, d_{22}\, d_{33}\, d_{44} \geq 0$.
A (1, 2, 3, 5)	Det A (1, 2, 3, 5) = $d_{11}\, d_{22}\, d_{33}\, d_{55} \geq 0$.
A (1, 2, 4, 5)	Det A (1, 2, 4, 5) = $d_{11}\, d_{22}\, d_{44}\, d_{55} \geq 0$.
A (1, 3, 4, 5)	Det A (1, 3, 4, 5) = $d_{11}\, d_{33}\, d_{44}\, d_{55} \geq 0$.
A (2, 3, 4, 5)	Det A (2, 3, 4, 5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Det A = $d_{11}\, d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Det A (1,2) = $d_{11}\, d_{22} - a_{12}\, x_{21} \geq 0$. Since, $a_{12}\, x_{21} = 0$ after assigning $x_{21} = 0$. Correspondingly,

Det A (2,3), Det A (3,4), Det A (3,5) ≥ 0. Since all the determinants are non-negative then

the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

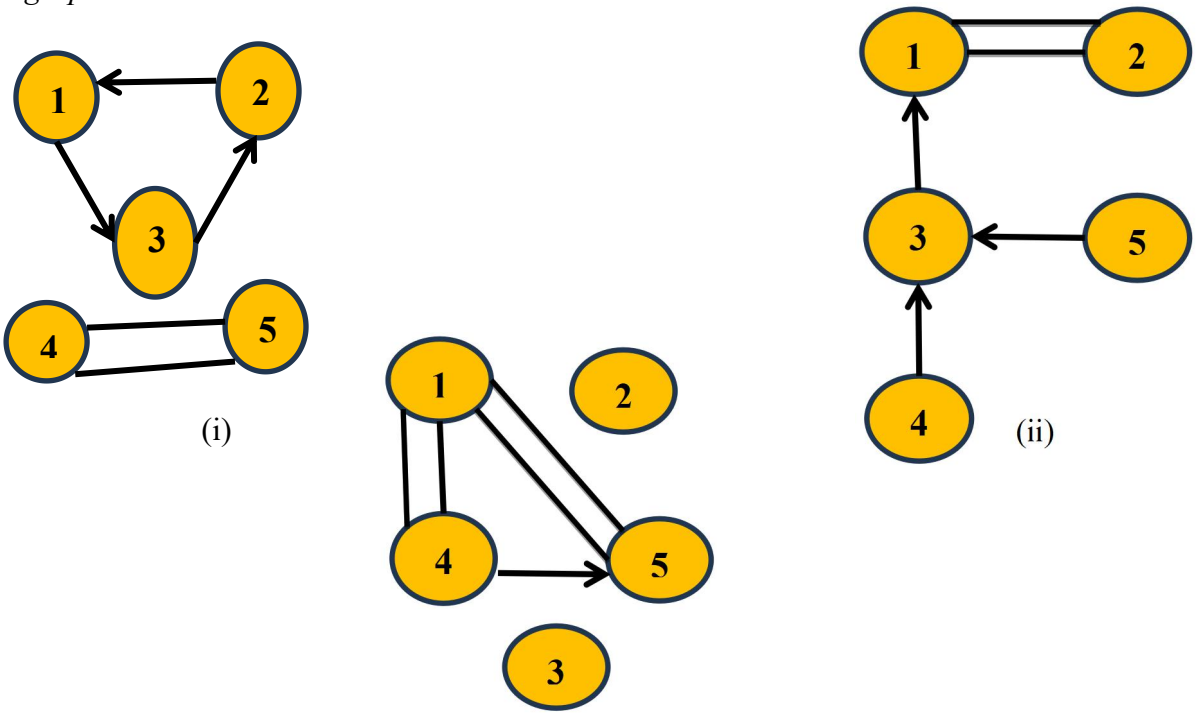
Therefore, digraph of order 5 with 4 arcs with positionally symmetric cycle was concluded not to admit zero completion into a Wss P_0-matrix. All the other digraphs which are cyclic or acyclic were observed to have zero completion into a Wss P_0-matrix.

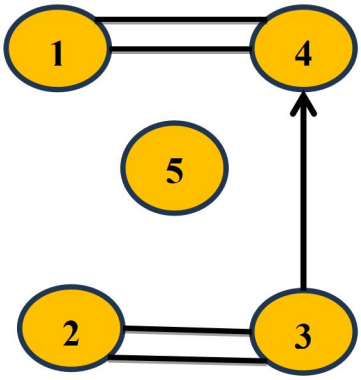
4.6 Digraphs of Order 5 with 5 Arcs

According to the technique developed by Harary *et al. (*2018), there are 154 non-isomorphic digraphs with 5 vertices and 5 arcs. These are few examples of digraphs with 5 vertices and 5 arcs.

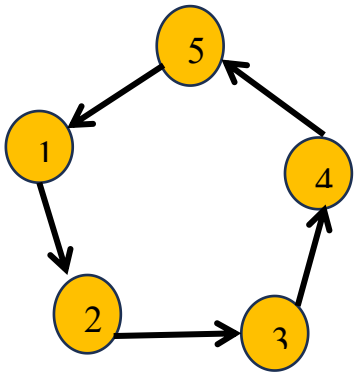
Figure 4. 12

Digraphs with 5 vertices and 5 arcs

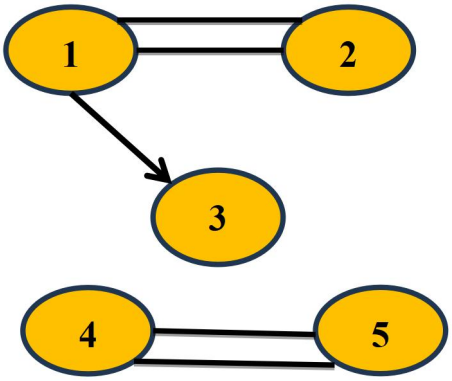




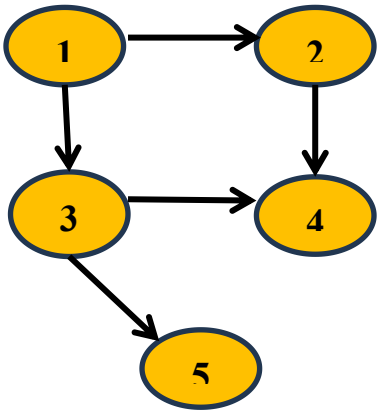
(iv)



(v)



(vi)



(vii)

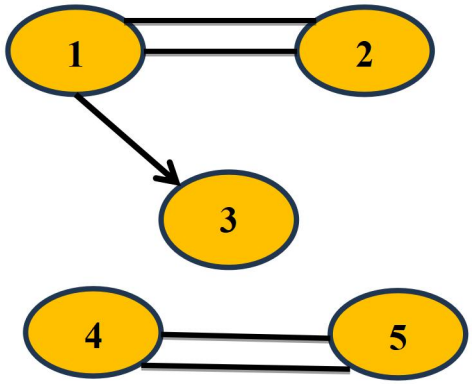
Source: Researcher (2024)

Case 1

Example 9

Figure 4. 13

A digraph with clique sub-digraph



Source: Researcher (2024)

The partial matrix that specifies the above digraph A is= $\begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & a_{54} & d_{55} \end{pmatrix}$

The other digraphs and their corresponding partial Wss P_0-matrices could be constructed and extracted respectively, in a similar manner.

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

A (1, 2) = $\begin{bmatrix} d_{11} & a_{12} \\ a_{21} & d_{22} \end{bmatrix}$, A (1, 3) = $\begin{bmatrix} d_{11} & a_{13} \\ x_{31} & d_{33} \end{bmatrix}$

A (1, 4) = $\begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}$, A (1, 5) = $\begin{bmatrix} d_{11} & x_{15} \\ x_{51} & d_{55} \end{bmatrix}$, A (2, 3) = $\begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$

A (2, 4) = $\begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$, A (2, 5) = $\begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, A (3,4) = $\begin{bmatrix} d_{33} & x_{34} \\ x_{43} & d_{44} \end{bmatrix}$

A (3,5) = $\begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}$, A (4, 5) = $\begin{bmatrix} d_{44} & a_{45} \\ a_{54} & d_{55} \end{bmatrix}$,

A(1,2,3) = $\begin{bmatrix} d_{11} & a_{12} & a_{13} \\ a_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}$, A(1,2,4) = $\begin{bmatrix} d_{11} & a_{12} & x_{14} \\ a_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}$, A(1,2,5) = $\begin{bmatrix} d_{11} & a_{12} & x_{15} \\ a_{21} & d_{22} & x_{25} \\ x_{51} & x_{52} & d_{55} \end{bmatrix}$

A(1,3,4) = $\begin{bmatrix} d_{11} & a_{13} & x_{14} \\ x_{31} & d_{33} & x_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix}$, A(1,3,5) = $\begin{bmatrix} d_{11} & a_{13} & x_{15} \\ x_{31} & d_{33} & x_{35} \\ x_{51} & x_{53} & d_{55} \end{bmatrix}$, A(1,4,5) = $\begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & x_{45} \\ x_{51} & x_{54} & d_{55} \end{bmatrix}$

A(2,3,4) = $\begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & x_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}$, A(2,3,5 = $\begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}$, A(2,4,5) = $\begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & x_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$

A (3,4,5) = $\begin{pmatrix} d_{33} & x_{34} & x_{35} \\ x_{43} & d_{44} & a_{45} \\ x_{53} & a_{54} & d_{55} \end{pmatrix}$, A (1,2,3,4) = $\begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} \\ a_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix}$,

A (1,2,3,5) = $\begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ x_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}$, A(1,2,4,5) = $\begin{pmatrix} d_{11} & a_{12} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & a_{45} \\ x_{51} & x_{52} & a_{54} & d_{55} \end{pmatrix}$

A(1,3,4,5) = $\begin{pmatrix} d_{11} & a_{13} & x_{14} & x_{15} \\ x_{31} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{53} & a_{54} & d_{55} \end{pmatrix}$, A(2,3,4,5) = $\begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & a_{45} \\ x_{52} & x_{53} & a_{54} & d_{55} \end{pmatrix}$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1,2) = $d_{11}\, d_{22} - a_{12}\, a_{21}$

Det A (1,3) = $d_{11}\, d_{33} - a_{13}\, x_{31}$

Det A (1,4) = $d_{11}\, d_{44} - x_{11}\, x_{41}$

Det A (1,5) = $d_{11}\, d_{55} - x_{15}\, x_{51}$

Det A (2,3) = $d_{22}\, d_{33} - x_{23}\, x_{32}$

Det A (2,4) = $d_{22}\, d_{44} - x_{24}\, x_{42}$

Det A (2,5) = $d_{22}\, d_{55} - x_{25}\, x_{52}$

Det A (3,4) = $d_{33}\, d_{44} - x_{34}\, x_{43}$

Det A (3,5) = $d_{33}\, d_{55} - x_{35}\, x_{53}$

Det A (4 5) = $d_{44}\, d_{55} - a_{45}\, a_{54}$

Det A (1,2,3) = $d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32}) - a_{12}\,(a_{21}\, d_{33} - x_{23}\, x_{31}) + a_{13}\,(a_{21}\, x_{32} - d_{22}\, x_{31})$

Det A (1,2,4) = $d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - a_{12}\,(a_{21}\, d_{44} - x_{24}\, x_{41}) + x_{14}\,(a_{21}\, x_{42} - d_{22}\, x_{41})$

Det A (1,2,5) = $d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52}) - a_{12}\,(a_{21}\, d_{55} - x_{25}\, x_{51}) + x_{15}\,(a_{21}\, x_{52} - d_{22}\, x_{51})$

Det A (1,3,4) = $d_{11}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - a_{13}\,(x_{31}\, d_{44} - x_{34}\, x_{41}) + x_{14}\,(x_{31}\, x_{43} - d_{33}\, x_{41})$

Det A (1,3,5) = $d_{11}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - a_{13}\,(x_{31}\, d_{55} - x_{35}\, x_{51}) + x_{15}\,(x_{31}\, x_{53} - d_{33}\, x_{51})$

Det A (1,4,5) = $d_{11}\,(d_{44}\, d_{55} - a_{45}\, a_{54}) - x_{14}\,(x_{41}\, d_{55} - a_{45}\, x_{51}) + x_{15}\,(x_{41}\, a_{54} - d_{44}\, x_{51})$

Det A (2,3,4) = $d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$

Det A (2,3,5) = $d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$

Det A (2,4,5) = $d_{22}\,(d_{44}\, d_{55} - a_{45}\, a_{54}) - x_{24}\,(x_{42}\, d_{55} - a_{45}\, x_{52}) + x_{25}\,(x_{42}\, a_{54} - d_{44}\, x_{52})$.

Det A (3,4,5) = $d_{33}\,(d_{44}\, d_{55} - a_{45}\, a_{54}) - x_{34}\,(x_{43}\, d_{55} - a_{45}\, x_{53}) + x_{35}\,(x_{43}\, a_{54} - d_{44}\, x_{53})$.

Det A (1,2,3,4) = $d_{11}[d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})]$

$- a_{12}\,[a_{21}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, x_{41})]$

$+ a_{13}\,[a_{21}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, x_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, x_{41})$

$- x_{14}\,[a_{21}\,(x_{32}\, x_{43} - d_{33}\, x_{42}) - d_{22}(x_{43}\, x_{31} - d_{33}\, x_{41}) + x_{23}\,(x_{31}\, x_{42} - x_{32}\, x_{41})]$.

Det A (1,2,3,5) = $d_{11}\,[d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{32} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})]$

$- a_{12}\,[a_{21}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{31} - x_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{53} - d_{33}\, x_{51})]$

$+ a_{13}\,[a_{21}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) - d_{22}\,(d_{55}\, x_{31} - x_{35}\, x_{51}) + x_{25}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$

$- x_{15}\,[a_{21}\,(x_{32}\, x_{53} - d_{33}\, x_{52}) - d_{22}\,(x_{53}\, x_{31} - d_{33}\, x_{51}) + x_{23}\,(x_{31}\, x_{52} - x_{32}\, x_{51})]$.

Det A (1,2,4,5) = $d_{11}\,[d_{22}\,(d_{44}\, d_{55} - a_{45}\, a_{54}) - x_{24}\,(d_{55}\, x_{42} - a_{45}\, x_{52}) + x_{25}\,(x_{42}\, a_{54} - d_{44}\, x_{52})]$

$- a_{12}\,[a_{21}\,(d_{44}\, d_{55} - a_{45}\, a_{54}) - x_{24}\,(d_{55}\, x_{41} - a_{45}\, x_{51}) + x_{25}\,(x_{41}\, a_{54} - d_{44}\, x_{51})]$

$$+ x_{14}\,[a_{21}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) - d_{22}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,x_{51})]$$

$$- x_{15}\,[a_{21}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) - d_{22}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{24}\,(x_{41}\,x_{52} - x_{51}\,x_{42})].$$

$$\text{Det A } (1,3,4,5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,x_{53})]$$

$$- a_{13}\,[x_{31}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51})]$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51})]$$

$$- x_{15}\,[x_{31}\,(x_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{51}\,x_{43})].$$

$$\text{Det A } (2,3,4,5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) - x_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52})].$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- x_{25}\,(x_{32}\,(x_{43}\,a_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))]$$

$$- a_{12}\,[a_{21}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,a_{54} - d_{44}\,x_{53}))$$

$$+ x_{23}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51}))$$

$$- x_{24}\,(x_{31}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$+ x_{25}\,(x_{31}\,(x_{43}\,a_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))]$$

$$+ a_{13}\,[a_{21}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,a_{54} - d_{44}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,a_{54}) - x_{34}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,a_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - a_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{42}\,a_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,a_{54} - d_{44}\,x_{51}) + x_{34}\,(x_{41}\,x_{52} - x_{42}\,x_{51}))]$$

$$- x_{14}\,[a_{21}\,(x_{32}(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$+d_{22}(x_{31}(d_{55}x_{43}-a_{45}x_{53})-d_{33}(d_{55}x_{41}-a_{45}x_{51})+x_{35}(x_{41}x_{53}-x_{43}x_{51}))$$

$$-x_{23}(x_{31}(d_{55}x_{42}-a_{45}x_{52})-x_{32}(d_{55}x_{41}-a_{45}x_{51})+x_{35}(x_{41}x_{52}-x_{42}x_{51}))$$

$$x_{25}(x_{31}(x_{53}x_{42}-x_{43}x_{52})-x_{32}(x_{41}x_{53}-x_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51}))]$$

$$+x_{15}[a_{21}(x_{32}(x_{43}a_{54}-d_{44}x_{53})-d_{33}(x_{42}a_{54}-d_{44}x_{52})+x_{34}(x_{42}x_{53}-x_{43}x_{52}))$$

$$-d_{22}(x_{31}(a_{54}x_{43}-d_{44}x_{53})-d_{33}(x_{41}a_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{53}-x_{43}x_{51}))$$

$$+x_{23}(x_{31}(x_{42}a_{54}-d_{44}x_{52})-x_{32}(x_{41}a_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{52}-x_{42}x_{51}))$$

$$-x_{24}(x_{31}(x_{42}x_{53}-x_{43}x_{52})-x_{32}(x_{41}x_{53}-x_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51}))].$$

All unknown entries X_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}=x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54}=0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 25

Determinants of 2×2 sub-matrices of " clique sub-digraph, order 5, 5 arc."

Principal sub-matrix	**Principal minor**
A (1,2)	Det A (1,2) = $(d_{11}\ d_{22}) - (a_{12}\ a_{21}) \geq 0$. (Since (1,2) is completely defined.
A (1,3)	Det A (1,3) = $d_{11}\ d_{33} \geq 0$
A (1,4)	Det A (1,4) = $d_{11}\ d_{44} \geq 0$
A (1,5)	Det A (1,5) = $d_{11}\ d_{55} \geq 0$
A (2,3)	Det A (2,3) = $d_{22}\ d_{33} \geq 0$
A (2,4)	Det A (2,4) = $d_{22}\ d_{44} \geq 0$
A (2,5)	Det A (2,5) = $d_{22}\ d_{55} \geq 0$
A(3,4)	Det A (3,4) = $d_{33}\ d_{44} \geq 0$
A (3,5)	Det A (3,5) = $d_{33}\ d_{55} \geq 0$
A (4,5)	Det A (4,5) = $(d_{44}\ d_{55} - a_{45}\ a_{54}) \geq 0$. (since (4,5) is completely defined).

Source: Researcher (2024)

Table 4. 26

Determinants of 3×3 sub-matrices of " clique sub-digraph, order 5, 5 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3)	Det A (1, 2, 3) = d_{33} (d_{11} d_{22} – a_{12} a_{21}) ≥ 0. (since (1, 2) is completely defined).
A (1,2,4)	Det A (1, 2, 4) = d_{44} (d_{11} d_{22} – a_{12} a_{21}) ≥ 0. (since (1,2) is completely defined).
A (1,2,5)	Det A (1, 2, 5) = d_{55} (d_{11} d_{22} – a_{12} a_{21}) ≥ 0. (since (1, 2) is completely defined).
A (1,3,4)	Det A (1, 3, 4) = d_{11} d_{33} d_{44} ≥ 0
A (1,3,5)	Det A (1, 3, 5) = d_{11} d_{33} d_{55} ≥ 0
A (1,4,5)	Det A (1, 4, 5) = d_{11} (d_{44} d_{55} – a_{45} a_{54}) ≥ 0. (since (4, 5) is completely defined).
A (2,3,4)	Det A (2, 3, 4) = d_{22} d_{33} d_{44} ≥ 0
A (2,3,5)	Det A (2, 3, 5) = d_{22} d_{33} d_{55} ≥ 0
A (2,4,5)	Det A (2, 4, 5) = d_{22} (d_{44} d_{55} – a_{45} a_{54}) ≥ 0. (since (4,5) is completely defined).
A (3,4,5)	Det A (3, 4, 5) = d_{33} (d_{44} d_{55} – a_{45} a_{54}) ≥ 0. (since (4,5) is completely defined).

Source: Researcher (2024)

Table 4. 27

Determinants of 4×4 sub-matrices of " clique sub-digraph, order 5, 5 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3,4)	Det A (1,2,3,4) =d_{33} d_{44} (d_{11} d_{22} – a_{12} a_{21}) ≥ 0. (since (1,2) is completely defined).
A (1,2,3,5)	Det A (1,2,3,5) = d_{33} d_{55} (d_{11} d_{22} – a_{12} a_{21}) ≥ 0. (since (1,2) is completely defined).
A (1,2,4,5)	Det A (1,2,4,5) = d_{44} d_{55} (d_{11} d_{22} – a_{12} a_{21}) – d_{11} d_{22} a_{45} a_{54}.
A (1,3,4,5)	Det A (1,3,4,5) = d_{11} d_{33} (d_{44} d_{55} – a_{45} a_{54}) ≥ 0. (since (4,5) is completely defined).
A (2,3,4,5)	Det A (2,3,4,5) = d_{22} d_{33}(d_{44} d_{55} – a_{45} a_{54}) ≥ 0. (since (4,5) is completely defined).

Source: Researcher (2024)

Det A = d_{11} d_{22} d_{33} d_{44} d_{55} – a_{12} a_{21} d_{33} (d_{44} d_{55} – a_{45} a_{54}). Since all the determinants are non-negative then the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

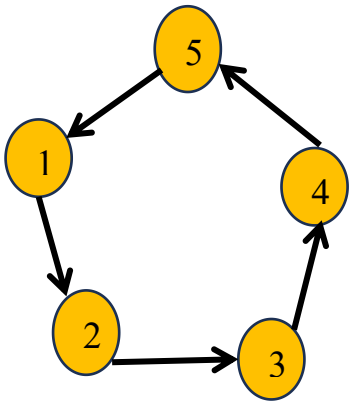
Case 2

Example 10

Assume digraph D = {(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4), (4,5), (5,1), (5,5)} with 5 vertices and 4 arcs given by:

Figure 4. 14

Cyclic digraph D of order 5 and 5 arcs



Source: Researcher (2024)

The matrix that partially outlines the above digraph is $A\begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

$$A(1,2) = \begin{bmatrix} d_{11} & a_{12} \\ x_{21} & d_{22} \end{bmatrix}, A(1,3) = \begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$$

$$A(1,4) = \begin{bmatrix} d_{11} & x_{14} \\ x_{41} & d_{44} \end{bmatrix}, A(1,5) = \begin{bmatrix} d_{11} & x_{15} \\ a_{51} & d_{55} \end{bmatrix}, A(2,3) = \begin{bmatrix} d_{22} & a_{23} \\ x_{32} & d_{33} \end{bmatrix}$$

$$A(2,4) = \begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}, A(2,5) = \begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}, A(3,4) = \begin{bmatrix} d_{33} & a_{34} \\ x_{43} & d_{44} \end{bmatrix}$$

$$A(3,5) = \begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}, A(4,5) = \begin{bmatrix} d_{44} & a_{45} \\ x_{54} & d_{55} \end{bmatrix},$$

$$A(1,2,3) = \begin{bmatrix} d_{11} & a_{12} & x_{13} \\ x_{21} & d_{22} & a_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}, A(1,2,4) = \begin{bmatrix} d_{11} & a_{12} & x_{14} \\ x_{21} & d_{22} & x_{24} \\ x_{41} & x_{42} & d_{44} \end{bmatrix}, A(1,2,5) = \begin{bmatrix} d_{11} & a_{12} & x_{15} \\ x_{21} & d_{22} & x_{25} \\ a_{51} & x_{52} & d_{55} \end{bmatrix}$$

$$A(1,3,4) = \begin{bmatrix} d_{11} & x_{13} & x_{14} \\ x_{31} & d_{33} & a_{34} \\ x_{41} & x_{43} & d_{44} \end{bmatrix}, A(1,3,5) = \begin{bmatrix} d_{11} & x_{13} & x_{15} \\ x_{31} & d_{33} & x_{35} \\ a_{51} & x_{53} & d_{55} \end{bmatrix}, A(1,4,5) = \begin{bmatrix} d_{11} & x_{14} & x_{15} \\ x_{41} & d_{44} & a_{45} \\ a_{51} & x_{54} & d_{55} \end{bmatrix}$$

$$A(2,3,4) = \begin{bmatrix} d_{22} & a_{23} & x_{24} \\ x_{32} & d_{33} & a_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}, A(2,3,5) = \begin{bmatrix} d_{22} & a_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}, A(2,4,5) = \begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & a_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$$

$$A(3,4,5) = \begin{pmatrix} d_{33} & a_{34} & x_{35} \\ x_{43} & d_{44} & x_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix}, A(1,2,3,4) = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} \\ x_{21} & d_{22} & a_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & a_{34} \\ x_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix},$$

$$A(1,2,3,5) = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ a_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}, A(1,2,4,5) = \begin{pmatrix} d_{11} & a_{12} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$$

$$A(1,3,4,5) = \begin{pmatrix} d_{11} & x_{13} & x_{14} & x_{15} \\ x_{31} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}, A(2,3,4,5) = \begin{pmatrix} d_{22} & a_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & a_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & a_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

Determinants of each sub-matrix obtained were obtained as indicated:

$$\text{Det } A(1,2) = d_{11}\, d_{22} - a_{12}\, x_{21}$$

$$\text{Det } A(1,3) = d_{11}\, d_{33} - x_{13}\, x_{31}$$

$$\text{Det } A(1,4) = d_{11}\, d_{44} - x_{14}\, x_{41}$$

$$\text{Det } A(1,5) = d_{11}\, d_{55} - x_{15}\, a_{51}$$

$$\text{Det } A(2,3) = d_{22}\, d_{33} - a_{23}\, x_{32}$$

$$\text{Det } A(2,4) = d_{22}\, d_{44} - x_{24}\, x_{42}$$

$$\text{Det } A(2,5) = d_{22}\, d_{55} - x_{25}\, x_{52}$$

$$\text{Det } A(3,4) = d_{33}\, d_{44} - a_{34}\, x_{43}$$

$$\text{Det } A(3,5) = d_{33}\, d_{55} - x_{35}\, x_{53}$$

$$\text{Det } A(4,5) = d_{44}\, d_{55} - a_{45}\, x_{54}$$

$$\text{Det } A(1,2,3) = d_{11}\,(d_{22}\, d_{33} - a_{23}\, x_{32}) - a_{12}\,(x_{21}\, d_{33} - a_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22}\, x_{31})$$

$$\text{Det A } (1, 2, 4) = d_{11}\,(d_{22}\,d_{44} - x_{24}\,x_{42}) - a_{12}\,(x_{21}\,d_{44} - x_{24}\,x_{41}) + x_{14}\,(x_{21}\,x_{42} - d_{22}\,x_{41})$$

$$\text{Det A } (1, 2, 5) = d_{11}\,(d_{22}\,d_{55} - x_{25}\,x_{52}) - a_{12}\,(x_{21}\,d_{55} - x_{25}\,a_{51}) + x_{15}\,(x_{21}\,x_{52} - d_{22}\,a_{51})$$

$$\text{Det A } (1, 3, 4) = d_{11}\,(d_{33}\,d_{44} - a_{34}\,x_{43}) - x_{13}\,(x_{31}\,d_{44} - a_{34}\,x_{41}) + x_{14}\,(x_{31}\,x_{43} - d_{33}\,x_{41})$$

$$\text{Det A } (1, 3, 5) = d_{11}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - x_{13}\,(x_{31}\,d_{55} - x_{35}\,a_{51}) + x_{15}\,(x_{31}\,x_{53} - d_{33}\,a_{51})$$

$$\text{Det A } (1, 4, 5) = d_{11}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{14}\,(x_{41}\,d_{55} - a_{45}\,a_{51}) + x_{15}\,(x_{41}\,x_{54} - d_{44}\,a_{51})$$

$$\text{Det A } (2, 3, 4) = d_{22}\,(d_{33}\,d_{44} - a_{34}\,x_{43}) - a_{23}\,(x_{32}\,d_{44} - a_{34}\,x_{42}) + x_{24}\,(x_{32}\,x_{43} - d_{33}\,x_{42})$$

$$\text{Det A } (2, 3, 5) = d_{22}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - a_{23}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) + x_{25}\,(x_{32}\,x_{53} - d_{33}\,x_{52})$$

$$\text{Det A } (2, 4, 5) = d_{22}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{24}\,(x_{42}\,d_{55} - a_{45}\,x_{52}) + x_{25}\,(x_{42}\,x_{54} - d_{44}\,x_{52}).$$

$$\text{Det A } (3, 4, 5) = d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}).$$

$$\begin{aligned}\text{Det A } (1, 2, 3, 4) = {} & d_{11}\,[d_{22}\,(d_{33}\,d_{44} - a_{34}\,x_{43}) - a_{23}\,(d_{44}\,x_{32} - a_{34}\,x_{42}) + x_{24}\,(x_{32}\,x_{43} - d_{33}\,x_{42})] \\ & - a_{12}\,[x_{21}\,(d_{33}\,d_{44} - a_{34}\,x_{43}) - a_{23}\,(d_{44}\,x_{31} - a_{34}\,x_{41}) + x_{24}\,(x_{31}\,x_{43} - d_{33}\,x_{41})] \\ & + x_{13}\,[x_{21}\,(x_{32}\,d_{44} - a_{34}\,x_{42}) - d_{22}\,(d_{44}\,x_{31} - a_{34}\,x_{41}) + x_{24}\,(x_{31}\,x_{42} - x_{32}\,x_{41})] \\ & - x_{14}\,[x_{21}\,(x_{32}\,x_{43} - d_{33}\,x_{42}) - d_{22}(x_{43}\,x_{31} - d_{33}\,x_{41}) + a_{23}\,(x_{31}\,x_{42} - x_{32}\,x_{41})].\end{aligned}$$

$$\begin{aligned}\text{Det A } (1, 2, 3, 5) = {} & d_{11}\,[d_{22}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - a_{23}\,(d_{55}\,x_{32} - x_{35}\,x_{52}) + x_{25}\,(x_{32}\,x_{53} - d_{33}\,x_{52})] \\ & - a_{12}\,[x_{21}\,(d_{33}\,d_{55} - x_{35}\,x_{53}) - a_{23}\,(d_{55}\,x_{31} - x_{35}\,a_{51}) + x_{25}\,(x_{31}\,x_{53} - d_{33}\,a_{51})] \\ & + x_{13}\,[x_{21}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) - d_{22}\,(d_{55}\,x_{31} - x_{35}\,a_{51}) + x_{25}\,(x_{31}\,x_{52} - x_{32}\,a_{51})] \\ & - x_{15}\,[x_{21}\,(x_{32}\,x_{53} - d_{33}\,x_{52}) - d_{22}\,(x_{53}\,x_{31} - d_{33}\,a_{51}) + a_{23}\,(x_{31}\,x_{52} - x_{32}\,a_{51})].\end{aligned}$$

$$\begin{aligned}\text{Det A } (1, 2, 4, 5) = {} & d_{11}\,[d_{22}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{24}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{25}\,(x_{42}\,x_{54} - d_{44}\,x_{52})] \\ & - a_{12}\,[x_{21}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{24}\,(d_{55}\,x_{41} - a_{45}\,a_{51}) + x_{25}\,(x_{41}\,x_{54} - d_{44}\,a_{51})] \\ & + x_{14}\,[x_{21}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) - d_{22}\,(d_{55}\,x_{41} - a_{45}\,a_{51}) + x_{25}\,(x_{41}\,x_{52} - x_{42}\,a_{51})] \\ & - x_{15}\,[x_{21}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - d_{22}\,(x_{41}\,x_{54} - d_{44}\,a_{51}) + x_{24}\,(x_{41}\,x_{52} - a_{51}\,x_{42})].\end{aligned}$$

$$\begin{aligned}\text{Det A } (1, 3, 4, 5) = {} & d_{11}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{5}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})] \\ & - x_{13}\,[x_{31}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{5}\,x_{41} - a_{45}\,a_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,a_{51})]\end{aligned}$$

$$+ x_{14}\,[x_{31}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_5\,x_{41} - a_{45}\,a_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,a_{51})]$$

$$- x_{15}\,[x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,a_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{51}\,x_{43})].$$

$$\text{Det A } (2, 3, 4, 5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- a_{23}\,[x_{32}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{25}\,[x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_3\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52})].$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- a_{23}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- x_{25}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))] -$$

$$- a_{12}\,[x_{21}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- \quad a_{23}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,x_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - a_{45}\,x_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,x_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{43}\,x_{51}))]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - a_{34}\,(d_{55}\,x_{41} - a_{45}\,a_{51}) + x_{35}\,(x_{41}\,x_{54} - d_{44}\,a_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - a_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - a_{45}\,a_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,a_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_4\,a_{51}) + a_{34}\,(x_{41}\,x_{52} - x_{42}\,a_{51}))]$$

$$- x_{14}\,[x_{21}\,(x_{32}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{41} - a_{45}\,a_{51}) + x_{35}\,(x_{41}\,x_{53} - x_{43}\,a_{51}))$$

$$+ a_{23}\,(x_{31}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) - x_{32}\,(d_{55}\,x_{41} - a_{45}\,a_{51}) + x_{35}\,(x_{41}\,x_{52} - x_{42}\,a_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{53}\,x_{42} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,a_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,a_{51}))]$$

$$+ x_{15}\,[x_{21}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + a_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(x_{54}\,x_{43} - d_{44}\,x_{53}) - d_{33}\,(x_{41}\,x_{54} - d_{44}\,a_{51}) + a_{34}\,(x_{41}\,x_{53} - x_{43}\,a_{51}))$$

$$+ a_{23}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(x_{41}\,x_{54} - d_{44}\,a_{51}) + a_{34}\,(x_{41}\,x_{52} - x_{42}\,a_{51}))$$

$$- x_{24}\,(x_{31}\,(x_{42}\,x_{53} - x_{43}\,x_{52}) - x_{32}\,(x_{41}\,x_{53} - x_{43}\,a_{51}) + d_{33}\,(x_{41}\,x_{52} - x_{42}\,a_{51}))].$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion,

$x_{12}=x_{13}= x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 28

Determinants of 2×2 sub-matrices of " cyclic digraph, order 5, 5 arc."

Principal sub-matrix	**Principal minor**
A (1, 2)	Det A (1, 2) = $d_{11}\,d_{22} \geq 0$.
A (1, 3)	Det A (1, 3) = $d_{11}\,d_{33} \geq 0$.
A (1, 4)	Det A (1, 4) = $d_{11}\,d_{44} \geq 0$
A (1, 5)	Det A (1, 5) = $d_{11}\,d_{55} \geq 0$
A (2, 3)	Det A (2, 3) = $d_{22}\,d_{33} \geq 0$
A (2, 4)	Det A (2, 4) = $d_{22}\,d_{44} \geq 0$
A (2, 5)	Det A (2, 5) = $d_{22}\,d_{55} \geq 0$
A (3, 4)	Det A (3 ,4) = $d_{33}\,d_{44} \geq 0$
A (3, 5)	Det A (3, 5) = $d_{33}\,d_{55} \geq 0$
A (4, 5)	Det A (4, 5) = $d_{44}\,d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 29

Determinants of 3×3 sub-matrices of " cyclic digraph, order 5, 5 arc."

Principal sub-matrix	**Principal minor**
A (1, 2, 3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$.
A (1, 2, 4)	Det A (1, 2, 4) = $d_{11}\, d_{22}\, d_{44} \geq 0$.
A (1, 2, 5)	Det A (1, 2, 5) = $d_{11}\, d_{22}\, d_{55} \geq 0$.
A (1, 3, 4)	Det A (1, 3, 4) = $d_{11}\, d_{33}\, d_{44} \geq 0$
A (1, 3, 5)	Det A (1, 3, 5) = $d_{11}\, d_{33}\, d_{55} \geq 0$
A (1, 4, 5)	Det A (1, 4, 5) = $d_{11}\, d_{44}\, d_{55} \geq 0$
A (2, 3, 4)	Det A (2, 3, 4) = $d_{22}\, d_{33}\, d_{44} \geq 0$
A (2, 3, 5)	Det A (2, 3, 5) = $d_{22}\, d_{33}\, d_{55} \geq 0$
A (2, 4, 5)	Det A (2, 4, 5) = $d_{22}\, d_{44}\, d_{55} \geq 0$
A (3, 4, 5)	Det A (3, 4, 5) = $d_{33}\, d_{44}\, d_{55} \geq 0$

Source: Researcher (2024)

Table 4. 30

Determinants of 4×4 sub-matrices of " cyclic digraph, order 5, 5 arc."

Principal sub-matrix	**Principal minor**
A (1, 2, 3, 4)	Det A (1, 2, 3, 4) = $d_{11}\, d_{22}\, d_{33}\, d_{44} \geq 0$.
A (1, 2, 3, 5)	Det A (1, 2, 3, 5) = $d_{11}\, d_{22}\, d_{33}\, d_{55} \geq 0$.
A (1, 2, 4, 5)	Det A (1, 2, 4, 5) = $d_{11}\, d_{22}\, d_{44}\, d_{55} \geq 0$.
A (1, 3, 4, 5)	Det A (1, 3, 4, 5) = $d_{11}\, d_{33}\, d_{44}\, d_{55} \geq 0$.
A (2, 3, 4, 5)	Det A (2, 3, 4, 5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Det A = $d_{11}\, d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Det A (1, 2) = $d_{11}\, d_{22} - a_{12}\, x_{21} \geq 0$. since, $a_{12}\, x_{21} = 0$ after assigning $x_{21} = 0$. Correspondingly,

Det A (2, 3), Det A (3, 4), Det A (3, 5) ≥ 0. Since all the determinants are non-negative then

the partial matrix can be completed into Wss P_0-matrix. Therefore, it was found to have zero completion into a Wss P_0-matrix.

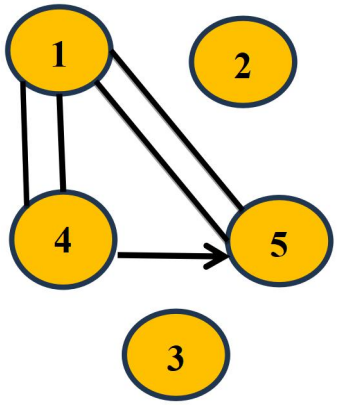
Case 3.

Example 11:

Assume digraph D = {(1,1), (1,4), (1,5), (2,2), (3,3), (3,4), (4,1), (4,4), (4,5), (5,1), (5,5)} with 5 vertices and 4 arcs given by:

Figure 4. 15

Digraph with clique sub-digraph



Source: Researcher (2024)

The partial matrix that specifies the above digraph A is= $\begin{pmatrix} d_{11} & x_{12} & x_{13} & a_{14} & a_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ a_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

As defined for partial Wss P_0-matrix $d_{11} \geq 0$, $d_{22} \geq 0$, $d_{33} \geq 0$, $d_{44} \geq 0$, $d_{55} \geq 0$.

All principal sub-matrices from the partial Wss P_0-matrix above were retrieved as shown below.

A (1, 2) = $\begin{bmatrix} d_{11} & x_{12} \\ x_{21} & d_{22} \end{bmatrix}$, A (1, 3) = $\begin{bmatrix} d_{11} & x_{13} \\ x_{31} & d_{33} \end{bmatrix}$

A (1, 4) = $\begin{bmatrix} d_{11} & a_{14} \\ a_{41} & d_{44} \end{bmatrix}$, A (1, 5) = $\begin{bmatrix} d_{11} & a_{15} \\ a_{51} & d_{55} \end{bmatrix}$, A (2, 3) = $\begin{bmatrix} d_{22} & x_{23} \\ x_{32} & d_{33} \end{bmatrix}$

A (2, 4) = $\begin{bmatrix} d_{22} & x_{24} \\ x_{42} & d_{44} \end{bmatrix}$, A (2, 5) = $\begin{bmatrix} d_{22} & x_{25} \\ x_{52} & d_{55} \end{bmatrix}$, A (3, 4) = $\begin{bmatrix} d_{33} & x_{34} \\ x_{43} & d_{44} \end{bmatrix}$

A (3, 5) = $\begin{bmatrix} d_{33} & x_{35} \\ x_{53} & d_{55} \end{bmatrix}$, A (4, 5) = $\begin{bmatrix} d_{44} & a_{45} \\ x_{54} & d_{55} \end{bmatrix}$,

A(1,2,3) = $\begin{bmatrix} d_{11} & x_{12} & x_{13} \\ x_{21} & d_{22} & x_{23} \\ x_{31} & x_{32} & d_{33} \end{bmatrix}$, A(1,2,4)= $\begin{bmatrix} d_{11} & x_{12} & a_{14} \\ x_{21} & d_{22} & x_{24} \\ a_{41} & x_{42} & d_{44} \end{bmatrix}$, A(1, 2, 5)= $\begin{bmatrix} d_{11} & x_{12} & a_{15} \\ x_{21} & d_{22} & x_{25} \\ a_{51} & x_{52} & d_{55} \end{bmatrix}$

A(1,3,4) = $\begin{bmatrix} d_{11} & x_{13} & a_{14} \\ x_{31} & d_{33} & x_{34} \\ a_{41} & x_{43} & d_{44} \end{bmatrix}$, A(1,3,5) = $\begin{bmatrix} d_{11} & x_{13} & a_{15} \\ x_{31} & d_{33} & x_{35} \\ a_{51} & x_{53} & d_{55} \end{bmatrix}$, A(1,4, 5) = $\begin{bmatrix} d_{11} & x_{14} & a_{15} \\ x_{41} & d_{44} & a_{45} \\ a_{51} & x_{54} & d_{55} \end{bmatrix}$

A(2,3,4)= $\begin{bmatrix} d_{22} & x_{23} & x_{24} \\ x_{32} & d_{33} & x_{34} \\ x_{42} & x_{43} & d_{44} \end{bmatrix}$, A(2,3,5) = $\begin{bmatrix} d_{22} & x_{23} & x_{25} \\ x_{32} & d_{33} & x_{35} \\ x_{52} & x_{53} & d_{55} \end{bmatrix}$, A (2,4,5) = $\begin{pmatrix} d_{22} & x_{24} & x_{25} \\ x_{42} & d_{44} & a_{45} \\ x_{52} & x_{54} & d_{55} \end{pmatrix}$

A (3, 4, 5) = $\begin{pmatrix} d_{33} & x_{34} & x_{35} \\ x_{43} & d_{44} & a_{45} \\ x_{53} & x_{54} & d_{55} \end{pmatrix}$, A (1, 2, 3, 4) = $\begin{pmatrix} d_{11} & x_{12} & x_{13} & a_{14} \\ x_{21} & d_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & d_{33} & x_{34} \\ a_{41} & x_{42} & x_{43} & d_{44} \end{pmatrix}$,

A (1, 2, 3, 5) = $\begin{pmatrix} d_{11} & x_{12} & x_{13} & a_{15} \\ x_{21} & d_{22} & x_{23} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{35} \\ a_{51} & x_{52} & x_{53} & d_{55} \end{pmatrix}$, A (1, 2, 4, 5) = $\begin{pmatrix} d_{11} & x_{12} & a_{14} & a_{15} \\ x_{21} & d_{22} & x_{24} & x_{25} \\ x_{41} & x_{42} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$

A (1, 3, 4, 5) = $\begin{pmatrix} d_{11} & x_{13} & a_{14} & a_{15} \\ x_{31} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{54} & d_{55} \end{pmatrix}$, A (2, 3, 4, 5) = $\begin{pmatrix} d_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & d_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & d_{44} & a_{45} \\ x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

Determinants of each sub-matrix obtained were obtained as indicated:

Det A (1, 2) = $d_{11}\, d_{22} - x_{12}\, x_{21}$

Det A (1, 3) = $d_{11}\, d_{33} - x_{13}\, x_{31}$

Det A (1, 4) = $d_{11}\, d_{44} - a_{14}\, a_{41}$

$$\text{Det A } (1, 5) = d_{11}\, d_{55} - a_{15}\, a_{51}$$

$$\text{Det A } (2, 3) = d_{22}\, d_{33} - x_{23}\, x_{32}$$

$$\text{Det A } (2, 4) = d_{22}\, d_{44} - x_{24}\, x_{42}$$

$$\text{Det A } (2, 5) = d_{22}\, d_{55} - x_{25}\, x_{52}$$

$$\text{Det A } (3, 4) = d_{33}\, d_{44} - x_{34}\, x_{43}$$

$$\text{Det A } (3, 5) = d_{33}\, d_{55} - x_{35}\, x_{53}$$

$$\text{Det A } (4, 5) = d_{44}\, d_{55} - a_{45}\, x_{54}$$

$$\text{Det A } (1, 2, 3) = d_{11}\,(d_{22}\, d_{33} - x_{23}\, x_{32}) - x_{12}\,(x_{21}\, d_{33} - x_{23}\, x_{31}) + x_{13}\,(x_{21}\, x_{32} - d_{22}\, x_{31})$$

$$\text{Det A } (1, 2, 4) = d_{11}\,(d_{22}\, d_{44} - x_{24}\, x_{42}) - x_{12}\,(x_{21}\, d_{44} - x_{24}\, a_{41}) + a_{14}\,(x_{21}\, x_{42} - d_{22}\, a_{41})$$

$$\text{Det A } (1, 2, 5) = d_{11}\,(d_{22}\, d_{55} - x_{25}\, x_{52}) - x_{12}\,(x_{21}\, d_{55} - x_{25}\, a_{51}) + a_{15}\,(x_{21}\, x_{52} - d_{22}\, a_{51})$$

$$\text{Det A } (1, 3, 4) = d_{11}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{13}\,(x_{31}\, d_{44} - x_{34}\, a_{41}) + a_{14}\,(x_{31}\, x_{43} - d_{33}\, a_{41})$$

$$\text{Det A } (1, 3, 5) = d_{11}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{13}\,(x_{31}\, d_{55} - x_{35}\, a_{51}) + a_{15}\,(x_{31}\, x_{53} - d_{33}\, a_{51})$$

$$\text{Det A } (1, 4, 5) = d_{11}\,(d_{44}\, d_{55} - a_{45}\, x_{54}) - a_{14}\,(x_{41}\, d_{55} - a_{45}\, a_{51}) + a_{15}\,(a_{41}\, x_{54} - d_{44}\, a_{51})$$

$$\text{Det A } (2, 3, 4) = d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(x_{32}\, d_{44} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})$$

$$\text{Det A } (2, 3, 5) = d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(x_{32}\, d_{55} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})$$

$$\text{Det A } (2, 4, 5) = d_{22}\,(d_{44}\, d_{55} - a_{45}\, x_{54}) - x_{24}\,(x_{42}\, d_{55} - a_{45}\, x_{52}) + x_{25}\,(x_{42}\, x_{54} - d_{44}\, x_{52}).$$

$$\text{Det A } (3, 4, 5) = d_{33}\,(d_{44}\, d_{55} - a_{45}\, x_{54}) - x_{34}\,(x_{43}\, d_{55} - a_{45}\, x_{53}) + x_{35}\,(x_{43}\, x_{54} - d_{44}\, x_{53}).$$

$$\begin{aligned}\text{Det A } (1, 2, 3, 4) = {} & d_{11}\,[d_{22}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{32} - x_{34}\, x_{42}) + x_{24}\,(x_{32}\, x_{43} - d_{33}\, x_{42})] \\ & - x_{12}\,[x_{21}\,(d_{33}\, d_{44} - x_{34}\, x_{43}) - x_{23}\,(d_{44}\, x_{31} - x_{34}\, a_{41}) + x_{24}\,(x_{31}\, x_{43} - d_{33}\, a_{41})] \\ & + x_{13}\,[x_{21}\,(x_{32}\, d_{44} - a_{34}\, x_{42}) - d_{22}\,(d_{44}\, x_{31} - x_{34}\, a_{41}) + x_{24}\,(x_{31}\, x_{42} - x_{32}\, a_{41})] \\ & - a_{14}\,[x_{21}\,(x_{32}\, x_{43} - d_{33}\, x_{42}) - d_{22}(x_{43}\, x_{31} - d_{33}\, a_{41}) + x_{23}\,(x_{31}\, x_{42} - x_{32}\, a_{41})].\end{aligned}$$

$$\begin{aligned}\text{Det A } (1, 2, 3, 5) = {} & d_{11}\,[d_{22}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{32} - x_{35}\, x_{52}) + x_{25}\,(x_{32}\, x_{53} - d_{33}\, x_{52})] \\ & - x_{12}\,[x_{21}\,(d_{33}\, d_{55} - x_{35}\, x_{53}) - x_{23}\,(d_{55}\, x_{31} - x_{35}\, a_{51}) + x_{25}\,(x_{31}\, x_{53} - d_{33}\, a_{51})]\end{aligned}$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,d_{55} - x_{35}\,x_{52}) - d_{22}\,(d_{55}\,x_{31} - x_{35}\,a_{51}) + x_{25}\,(x_{31}\,x_{52} - x_{32}\,a_{51})]$$

$$- a_{15}\,[x_{21}\,(x_{32}\,x_{53} - d_{33}\,x_{52}) - d_{22}\,(x_{53}\,x_{31} - d_{33}\,a_{51}) + x_{23}\,(x_{31}\,x_{52} - x_{32}\,a_{51})].$$

$$\text{Det A } (1, 2, 4, 5) = d_{11}\,[d_{22}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{24}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{25}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$- x_{12}\,[x_{21}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{24}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{25}\,(a_{41}\,x_{54} - d_{44}\,a_{51})]$$

$$+ a_{14}\,[x_{21}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) - d_{22}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{25}\,(a_{41}\,x_{52} - x_{42}\,a_{51})]$$

$$- a_{15}\,[x_{21}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - d_{22}\,(a_{41}\,x_{54} - d_{44}\,a_{51}) + x_{24}\,(a_{41}\,x_{52} - a_{51}\,x_{42})].$$

$$\text{Det A } (1, 3, 4, 5) = d_{11}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{5}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{13}\,[x_{31}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{54} - d_{44}\,a_{51})]$$

$$+ a_{14}\,[x_{31}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{53} - x_{43}\,a_{51})]$$

$$- a_{15}\,[x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(a_{41}\,x_{54} - d_{44}\,a_{51}) + x_{34}\,(a_{41}\,x_{53} - a_{51}\,x_{43})].$$

$$\text{Det A } (2, 3, 4, 5) = d_{22}\,[d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53})]$$

$$- x_{23}\,[x_{32}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52})]$$

$$+ x_{24}\,[x_{32}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52})]$$

$$- x_{25}\,[x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{3}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52})].$$

$$\text{Det A} = d_{11}\,[d_{22}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- x_{23}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$+ x_{24}\,(x_{32}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- x_{25}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))]$$

$$- x_{12}\,[x_{21}\,(d_{33}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) + x_{35}\,(x_{43}\,x_{54} - d_{44}\,x_{53}))$$

$$- \; x_{23}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{54} - d_{44}\,a_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{43}\,d_{55} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{53} - x_{43}\,a_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(a_{41}\,x_{54} - d_{44}\,a_{51}) + a_{34}\,(a_{41}\,x_{53} - x_{43}\,a_{51}))]$$

$$+ x_{13}\,[x_{21}\,(x_{32}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{54} - d_{44}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{44}\,d_{55} - a_{45}\,x_{54}) - x_{34}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{54} - d_{44}\,a_{51}))$$

$$+ x_{24}\,(x_{31}\,(x_{42}\,d_{55} - a_{45}\,x_{52}) - x_{32}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{52} - x_{42}\,a_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(a_{41}\,x_{54} - d_{44}\,a_{51}) + a_{34}\,(a_{41}\,x_{52} - x_{42}\,a_{51}))]$$

$$- a_{14}\,[x_{21}\,(x_{32}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) + x_{35}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(d_{55}\,x_{43} - a_{45}\,x_{53}) - d_{33}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{53} - x_{43}\,a_{51}))$$

$$+ x_{23}\,(x_{31}\,(d_{55}\,x_{42} - a_{45}\,x_{52}) - x_{32}\,(d_{55}\,a_{41} - a_{45}\,a_{51}) + x_{35}\,(a_{41}\,x_{52} - x_{42}\,a_{51}))$$

$$- x_{25}\,(x_{31}\,(x_{53}\,x_{42} - x_{43}\,x_{52}) - x_{32}\,(a_{41}\,x_{53} - x_{43}\,a_{51}) + d_{33}\,(a_{41}\,x_{52} - x_{42}\,a_{51}))]$$

$$+ a_{15}\,[x_{21}\,(x_{32}\,(x_{43}\,x_{54} - d_{44}\,x_{53}) - d_{33}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) + x_{34}\,(x_{42}\,x_{53} - x_{43}\,x_{52}))$$

$$- d_{22}\,(x_{31}\,(x_{54}\,x_{43} - d_{44}\,x_{53}) - d_{33}\,(a_{41}\,x_{54} - d_{44}\,a_{51}) + x_{34}\,(a_{41}\,x_{53} - x_{43}\,a_{51}))$$

$$+ x_{23}\,(x_{31}\,(x_{42}\,x_{54} - d_{44}\,x_{52}) - x_{32}\,(a_{41}\,x_{54} - d_{44}\,a_{51}) + x_{34}\,(a_{41}\,x_{52} - x_{42}\,a_{51}))$$

$$- x_{24}\,(x_{31}\,(x_{42}\,x_{53} - x_{43}\,x_{52}) - x_{32}\,(a_{41}\,x_{53} - x_{43}\,a_{51}) + d_{33}\,(a_{41}\,x_{52} - x_{42}\,a_{51}))].$$

All unknown entries x_{ij} of A are assigned to 0, on Performing zero completion.

$x_{12}=x_{13}= x_{14}=x_{15}=x_{21}=x_{23}=x_{24}=x_{25}=x_{31}=x_{32}=x_{34}=x_{35}=x_{41}=x_{42}=x_{43}=x_{51}=x_{52}=x_{53}=x_{54} = 0.$

The determinants, which are the principal minors, are indicated in the tables below:

Table 4. 31

Determinants of 2×2 sub-matrices of " clique digraph, order 5, 5 arc."

Principal submatrix	**Principal minor**
A (1,2)	Det A (1,2) = $d_{11}\, d_{22} \geq 0$.
A (1,3)	Det A (1,3) = $d_{11}\, d_{33} \geq 0$.
A (1,4)	Det A (1,4) = $(d_{11}\, d_{44}) - (a_{14}\, a_{41}) \geq 0$. Since (1,4) is completely defined.
A (1,5)	Det A (1,5) = $(d_{11}\, d_{55}) - (a_{15}\, a_{51}) \geq 0$. Since (1,5) is completely defined.
A (2,3)	Det A (2,3) = $d_{22}\, d_{33} \geq 0$.
A (2,4)	Det A (2,4) = $d_{22}\, d_{44} \geq 0$.
A (2,5)	Det A (2,5) = $d_{22}\, d_{55} \geq 0$.
A (3,4)	Det A (3,4) = $d_{33}\, d_{44} \geq 0$.
A (3,5)	Det A (3,5) = $d_{33}\, d_{55} \geq 0$.
A (4,5)	Det A (4,5) = $d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 32

Determinants of 3×3 sub-matrices of " clique digraph, order 5, 5 arc."

Principal submatrix	**Principal minor**
A (1,2,3)	Det A (1,2,3) = $d_{11}\, d_{22}\, d_{33} \geq 0$.
A (1,2,4)	Det A (1,2,4) = $d_{22}\, (d_{11}\, d_{44} - a_{14}\, a_{41}) \geq 0$. Since (1,4) is completely defined.
A (1,2,5)	Det A (1,2,5) = $d_{22}\, (d_{11}\, d_{55} - a_{15}\, a_{51}) \geq 0$. Since (1,5) is completely defined.
A (1,3,4)	Det A (1,3,4) = $d_{33}\, (d_{11}\, d_{44} - a_{14}\, a_{41}) \geq 0$. Since (1,4) is completely defined.
A (1,3,5)	Det A (1,3,5) = $d_{33}\, (d_{11}\, d_{55} - a_{15}\, a_{51}) \geq 0$. Since (1,5) is completely defined.
A (1,4,5)	Det A (1,4,5) = $d_{11}\, d_{44}\, d_{55} - a_{14}\, a_{41}\, d_{55} + a_{14}\, a_{45}\, a_{51} - a_{15}\, a_{51}\, d_{44}$.
A (2,3,4)	Det A (2,3,4) = $d_{22}\, d_{33}\, d_{44} \geq 0$.
A (2,3,5)	Det A (2,3,5) =$d_{22}\, d_{33}\, d_{55} \geq 0$.
A (2,4,5)	Det A (2,4,5) = $d_{22}\, d_{44}\, d_{55} \geq 0$.
A (3,4,5)	Det A (3,4,5) = $d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Table 4. 33

Determinants of 4×4 sub-matrices of " clique digraph, order 5, 5 arc."

Principal sub-matrix	**Principal minor**
A (1,2,3,4)	Det A (1,2,3,4) = $d_{22}\, d_{33}\, (d_{11}\, d_{44} - a_{14}\, a_{41}) \geq 0$. Since (1,4) is completely defined.
A (1,2,3,5)	Det A (1,2,3,5) = $d_{22}\, d_{33}\, (d_{11}\, d_{55} - a_{15}\, a_{51}) \geq 0$. Since (1,5) is completely defined.
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}\, d_{22}\, d_{44}\, d_{55} - a_{14}\, a_{41}\, d_{22}\, d_{55} + a_{14}\, a_{45}\, d_{22}\, a_{51} - a_{15}\, a_{51}\, d_{22}\, d_{44}$.
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}\, d_{33}\, d_{44}\, d_{55} - a_{14}\, a_{41}\, d_{33}\, d_{55} + a_{14}\, a_{45}\, d_{33}\, a_{51} - a_{15}\, a_{51}\, d_{33}\, d_{44}$.
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}\, d_{33}\, d_{44}\, d_{55} \geq 0$.

Source: Researcher (2024)

Det A = $d_{11}\, d_{22}\, d_{33}\, d_{44}\, d_{55} + a_{14}\, a_{41}\, d_{22}\, d_{33}\, d_{55} - a_{14}\, a_{51}\, a_{45}\, d_{22}\, d_{33} \geq 0$.

Since not all the determinants are non-negative, the partial matrix cannot be completed into a Wss P_0-matrix. Therefore, it was found not to admit zero completion into a Wss P_0-matrix.

Counter example

Let the partial matrix specifying the sub digraph (1,4,5) be

A = $\begin{bmatrix} 2 & -3 & 3 \\ -2 & 4 & 1 \\ 4 & x_{54} & 6 \end{bmatrix}$. After substituting the unspecified entries with zero i.e. $x_{54} = 0$. Then

|A| = $d_{11}\, d_{44}\, d_{55} - a_{14}\, a_{41}\, d_{55} + a_{14}\, a_{45}\, a_{51} - a_{15}\, d_{44}\, a_{51}$

$= 48 – 36 – 12 – 48 = – 48 < 0$. Hence A (1,4, 5) has no completion.

All digraphs considered above are non-isomorphic.

Therefore, digraph of order 5 with 5 arcs with positionally symmetric cycle was concluded not to admit zero completion into a Wss P_0-matrix. All the other digraphs which are cyclic or acyclic were observed to have zero completion into a Wss P_0-matrix.

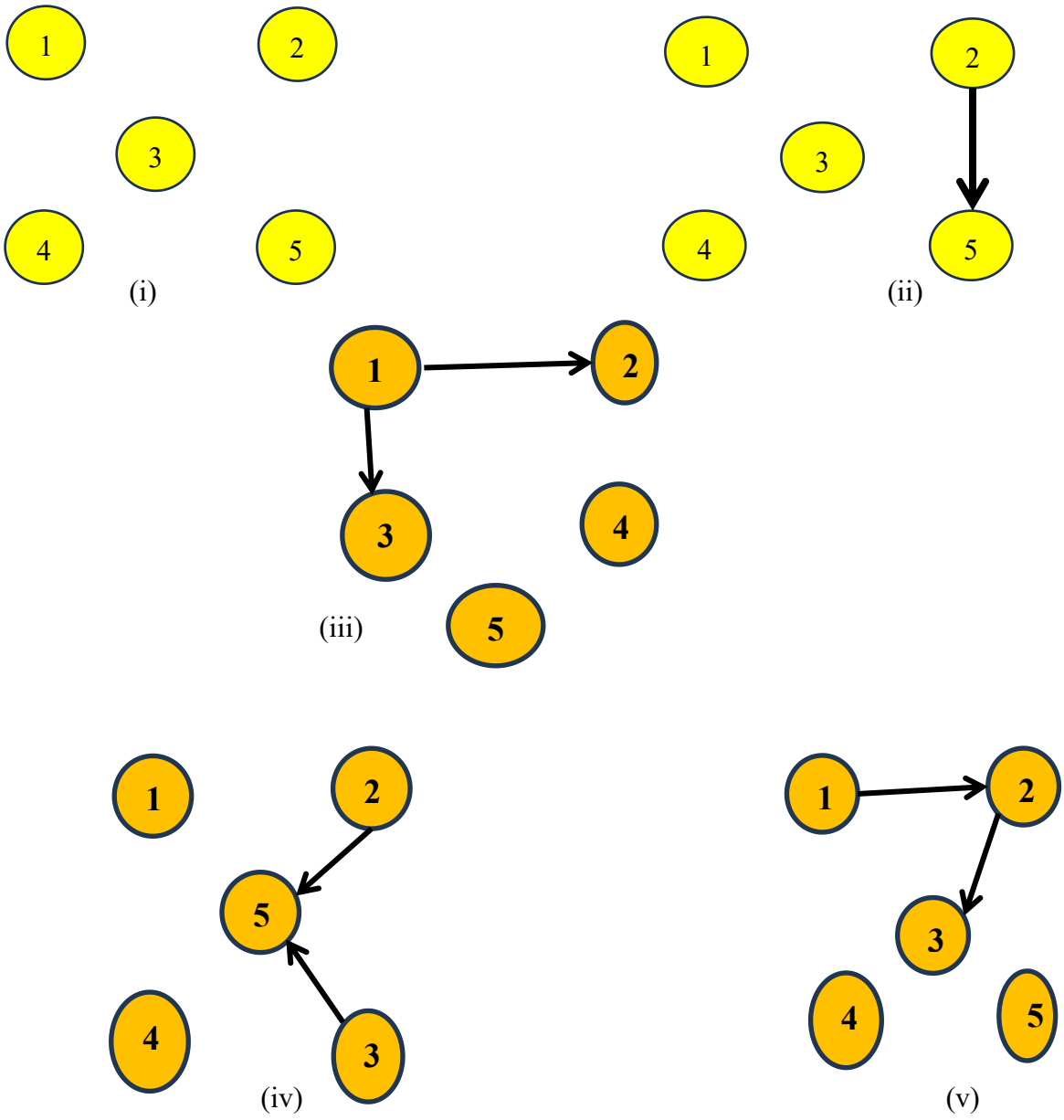
CHAPTER FIVE: SUMMARY, CONCLUSION, RECOMMENDATION AND PUBLICATION

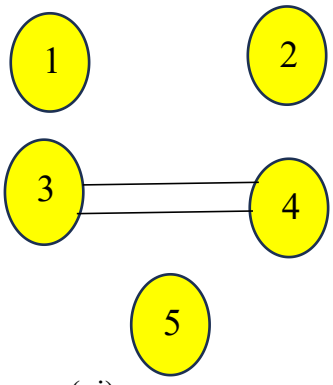
5.1 Summary

The study involved Wss P_0-matrices completion for digraphs of order 5 with up to 5 arcs. Partial matrix was extracted from each digraph and zero completion applied. With entries following a specific format. The diagonal entries were denoted by d_{ii}, while a_{ij} represents entries in the partial matrix corresponding to the directed edge ij in the digraph. Missing entries were marked as x_{ij}, indicating the absence of the directed edge ij in the digraph. The study showed that all non-isomorphic digraphs of order 5 with 0,1,2,3, arcs have completion.

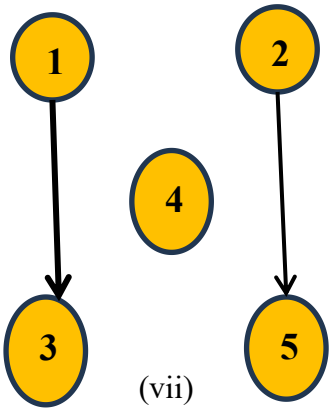
Figure 5. 1

Non-isomorphic digraphs of order 5 with 0,1,2,3, arcs

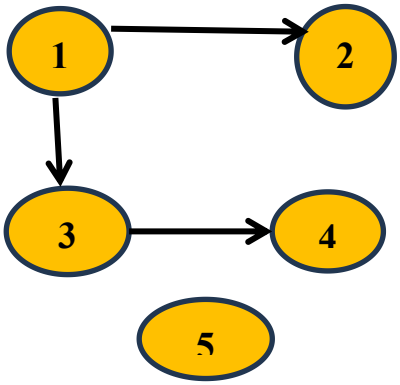




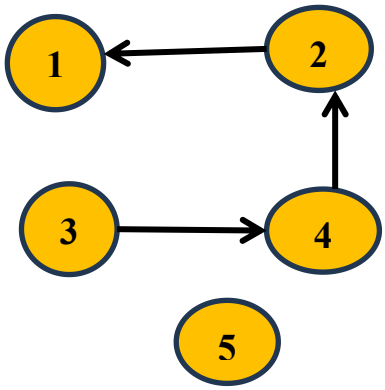
(vi)



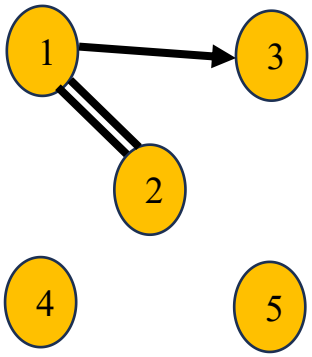
(vii)



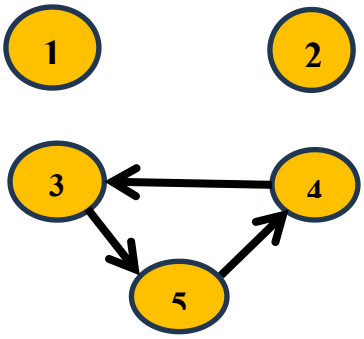
(viii)



(ix)



(x)



(xi)

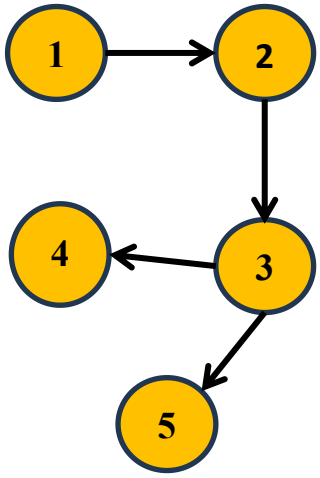
Source: Researcher (2025)

All the non-isomorphic digraphs above of order 5 with 0,1,2,3, arcs. All of them were observed to have completion into Wss P_0-matrix.

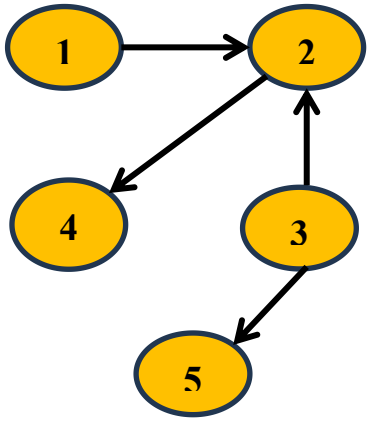
Below are Cyclic and acyclic Digraphs of order 5 with 4 and 5 arcs. Completion was applied on them and all were found to have zero completion into Wss P_0-matrix.

Figure 5. 2

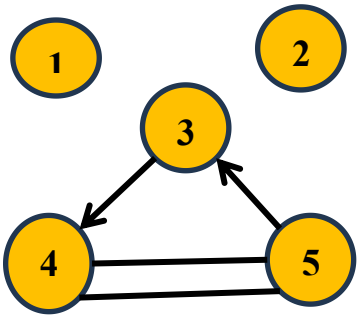
Cyclic and acyclic Digraphs of order 5 with 4 and 5 arcs



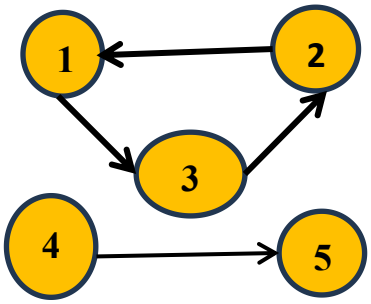
(i)



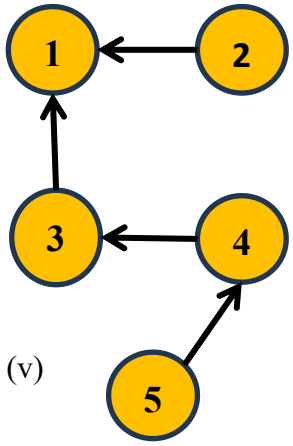
(ii)



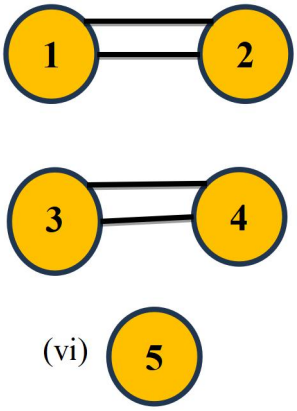
(iii)



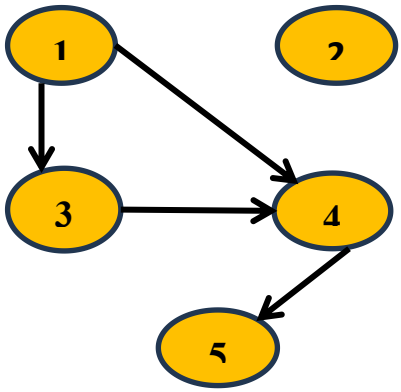
(iv)



(v)



(vi)



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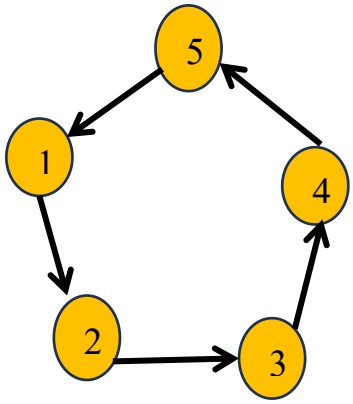

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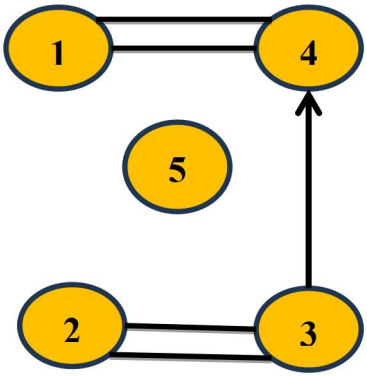

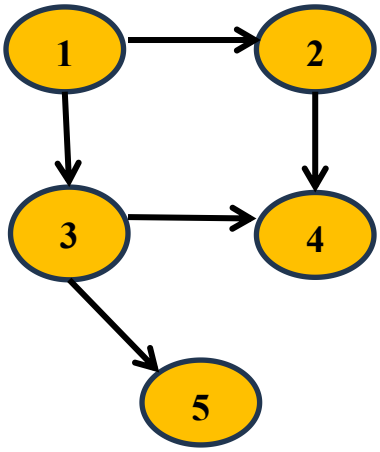

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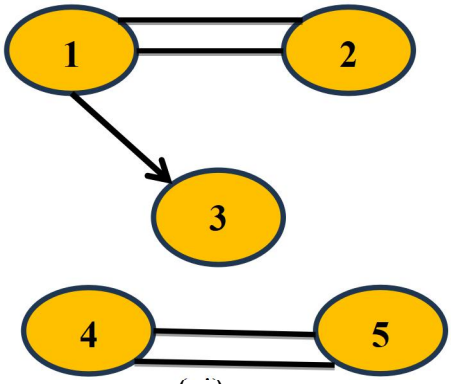
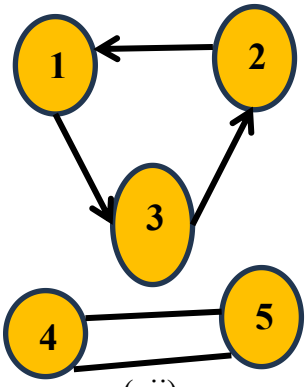


(xi)

(x)



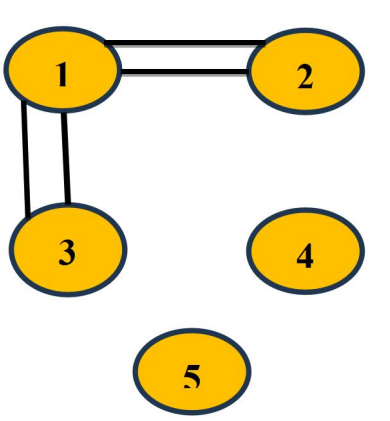
(xii)

Source: Researcher (2025)

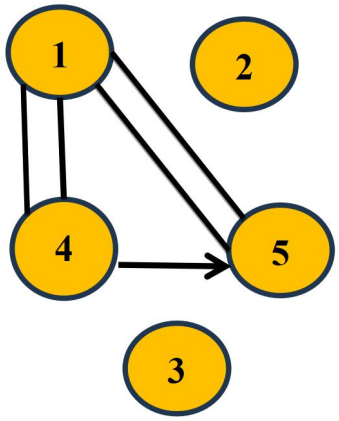
However, those digraphs of order 5 with 4 and 5 arcs with positionally symmetric cycle were discovered to have no completion.

Figure 5. 3

Digraphs of order 5 with 4 and 5 arcs with positionally symmetric cycle



(i)



(ii)

Source: Researcher (2025)

The digraphs above are of order 5 with 4 and 5 arcs respectively. Both of them have positionally symmetric cycle and were observed to have no completion into weakly sign symmetric P_0-matrix.

Digraphs of order 5 with 5 arcs with positionally symmetric cycle were found to inherit the digraph characteristics in digraphs of order 5 with 4 arcs which specifies positionally symmetric cycle. Whenever there is an order 5 with 4 arcs digraph with positionally symmetric cycle which lacks completion, the digraph with 5 arcs with the same cycle was observed not to have completion.

5.2 Conclusion

Upon comprehensive examination and analysis of various digraph patterns, the research arrived at following significant conclusions;

Partial matrices were derived from each digraph. Completion was applied on partial matrices determining their completion or non-completion. The characteristics of digraphs with completion or non-completion were determined. Cyclic and acyclic digraphs of order 5 with up to 5 arcs was found to have zero completion into Wss P_0-matrix. Digraphs of order 5 with 2 arcs which have positionally symmetric cycle were found to have completion. However, those digraphs of order 5 with 4 and 5 arcs which possess positionally symmetric cycle were discovered to have no completion.

5.3 Recommendation

Based on the findings of this study, the following recommendations for further research are made:

This study focused on the completion of partial weakly sign symmetric P_0-matrices defining digraphs of order 5 with up to 5 arcs. Future research could be done on other digraphs characteristics which lead to non-completion.

Future research can be carried out to come up with a computational code that can generate the determinants of the digraphs of high order and determine appropriate applications for completion of Wss P_0-matrices.

5.4 Publication

Marro, J., Mutembei, J., & Njagi, L. (2024). Completion of Weakly Sign Symmetric Po-matrix Problem for 5× 5 Matrices Specifying Digraphs of Order 5 with UP to 5 Arcs. *Asian Research Journal of Mathematics*, *20*(9), 1-14. https://doi.org/10.9734/arjom/2024/v20i9823

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APPENDICES

Appendix A: Journal Article

Asian Research Journal of Mathematics

Volume 20, Issue 9, Page 1-14, 2024; Article no.ARJOM.121188
ISSN: 2456-477X

Completion of Weakly Sign Symmetric Po-Matrix Problem for 5 × 5 Matrices Specifying Digraphs of Order 5 with UP to 5 Arcs

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Original Research Article

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Abstract

An n × n matrix is a weakly sign symmetric matrix if the off-diagonal elements have the property that if the entry in row i and column j is non-zero, then the entry in row j and column i must have same sign or zero. A digraph D has a Wss P_0 -matrix completion if every partial weakly sign symmetric P_0 -matrix that describes D can be extended to a complete weakly sign symmetric P_0 -matrix. This paper investigates the problem of completing weakly sign symmetric P_0 -matrices. It demonstrates that partial matrices representing all directed graphs of order 5 with edge strengths from 0 to 5 can indeed be completed to a weakly sign symmetric P_0 -matrix. Moreover, we established digraph characteristics that the partial Wss P_0- matrices specifying digraphs of order 5 with up to 5 arcs which have a clique and are cyclic or acyclic have zero completion into a Wss P_0-

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Appendix B: Plagiarism Report

DrillBit

The Report is Generated by DrillBit Plagiarism Detection Software

Submission Information

Author Name	JOSEPH LMAMBARIN MARRO
Title	WEAKLY SIGN SYMMETRIC P 0 -MATRIX COMPLETION PROBLEM FOR PATTERNS OF DIGRAPHS OF ORDER 5 WITH UP TO 5 ARCS
Paper/Submission ID	3928570
Submitted by	jwmaina@zimst.ac.ke
Submission Date	2025-06-23 12:05:45
Total Pages, Total Words	88, 38441
Document type	Thesis

Result Information

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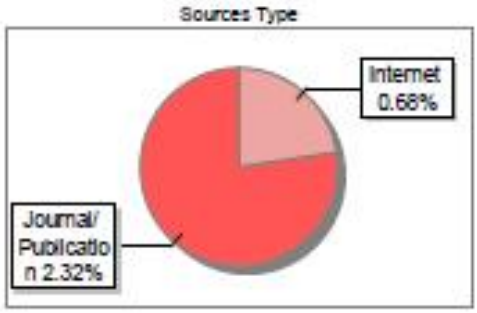

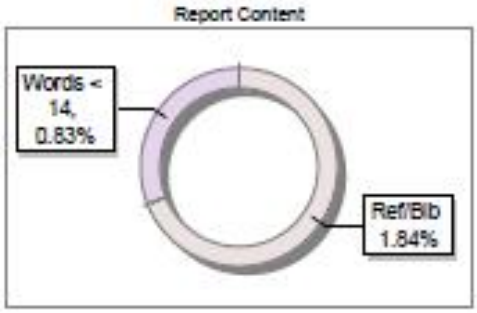

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